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A Brief History of Envelope Theorems in Economics: Static and Dynamic

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Abstract: This paper studies how envelope theorems have been used in Economics, their history and also who first introduced them. The existing literature is full of them and the reason is that most families of optimal value functions can produce them. The paper is driven by curiosity, but hopefully it will give the reader some new insights.

Keywords: Envelope theorems, names and history**,** value functions and cost-benefit analyses.

JEL-Codes: B16, B 21, B40

1. Introduction

Given that what we know about envelope theorems today in economics the (engineering) proofs are not difficult, but this was not true at the time they were known in economics from result by among others Hotelling (1932), Roy (1947) and Shephard (1953). It is obvious how Roy and Shephard came up with their results, but I used to tell graduate students that I will let them pass the microeconomics exam if they can find Hotelling's lemma in his article from (1932)[2]. The results may be viewed as corollaries of a general envelope theorem produced in mathematics. Mathematically an envelope is (loosely) defined as a curve that is touched by all members of a family of curves. There are theorems that in calculus give conditions for the existence of envelops to families of curves.

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[1] With assistance from Professor Erwin Diewert. Professor Thomas Aronsson Department of Economics, Umeå University and Professor Rolf Färe, Department of Economics, Oregon University, Corvallis commented previous versions of the document. They certainly improved the paper.

[2] The result can be found on page 22 in this paper..Paul Samuelson (1947) cites Hotelling (1932) without mentioning his "envelope result". This indicates that it may be a two pipe problem.

Some of the first envelope theorems produced by pure mathematicians may have been introduced by Ernst Zermelo (1894), Jean Darboux (1894) and Adolf Kneser (1898). They produced them in connection with new results in the calculus of variations.

The envelope theorem in calculus stands on its own, but the geometry is interesting for economic theory. It is well known that economists like Jacob Viner (1931), Roy Harrod (1931) and Erich Schneider (1931) used envelope properties to discuss the connection between short run and long run cost curves. Paul Samuelson (1947) derives the formal general proof of what today is called an envelope theorem, but under the headline "Displacement of Quantity Maximized"[3]. He mentions Viner's application of it as an example. Viner had a draftsman that produced his graph called Dr Wong. He insisted on tangency between the long run envelope cost curve and the short run curve, and he was right. However, he was not able to convince Viner. This means that there is a well- known error where a falling long run cost curve passes through the minimum of a short run cost curve.

Samuelson probably believed, at the time he produced his version of the envelope theorem, that he was the first to show what the second order change looked like, how the difference between the full second order change with respect to a parameter looked like in relation to the partial second order change, and how this difference could be signed by using the second order conditions (a negative semi-definite quadratic form for maximum). The last result is the only one that was new.

The first results in economics [4] on "comparative dynamics" in optimal control I have seen are available in a deep, not easy to read, paper by Oniki (1973), and they are based on the assumptions concerning the optimal control as a function of parameters. A proof of a special case appears in Benveniste and Scheinkman (1979). Seierstad (1981, 1982) proved under what conditions the (sub)-derivatives of the optimal value function exist and what they look like with respect to changes in the initial and final conditions and changes in parameters.

[3] Samuelson (1947) pp 34-35.

[4] There is also a result by Arrow in Arrow and Kurz (1970) based on dynamic programming that shows that the derivative of the value function with respect to initial conditions, calculated along an optimal path, is the adjoint function of the maximum principle.

When concavity is added, sub-derivatives change to derivatives. Slightly more general results were produced by Malanowski[5]in (1984).

There are also, eight and nine years later, two papers in the same journal as Seierstad's (1982) paper on derivatives of the value function with respect to parameters. The papers are written by Caputo (1990b) and La France and Barney (1991). They contain similar stuff although Seierstad's paper is the more stringent. Unlike Caputo and La France and Barney, Seierstad did not focus on derivatives with respect to parameters, but as we will see below parameters can be looked upon as "petrified" state variables.

With respect to the Calculus of Variations Caputo (1990a) has also contributed a paper on comparative dynamics via envelope methods. I am not sure that there exist many similar papers in the literature, but there exist a similar envelope result in my lecture notes that was copied from lectures by my teacher Tönu Puiu in the late seventies. Caputo seems, however, not to have seen Seierstad and Sydsaeter (1987) who have contributed with complete proofs of the differentiability of the optimal value function with respect to initial and final conditions and endpoint time[6].

2. The Austrian outlaws and the envelope theorem in economics

In this section, we will show how the envelope theorem may first have been introduced by economists rather than pure mathematicians. The two who did it were two Austrian cousins, Rudolp Auspitz and Richard Lieben, who, as Niehans (1990) writes," succeeded where Menger had failed, namely in providing the theory of price with an analytical apparatus". Both cousins were born in Vienna and both died there, but none of them belonged to the Viennese School which was dominated by among others Carl Menger, Eugen von Böhm-Bawerk , Fridriech Wieser and Gustav Schmoller. While Menger and others were occupied by "Der Metodenstreit", the outsiders Auzspitz and Lieben produced the only Austrian 19;th century contribution to mathematical economics; one of the outstanding contributions during the last two decades of the century . Both of them had studied mathematics. Auspitz did not finish his degree. He moved into business and founded one of the first sugar

[5] See also the references therein.

[6] See Chapter 1, where the results are produced as exercises for the reader. The mathematical textbooks I have consulted are Widder (1961), Cournot and John (volume 2 1974) and Rudin (1976). The latter did not mention envelopes.

refineries in Austria only 26 years old. After studying mathematics and engineering sciences Lieben also moved into business as a banker. As amateurs they produced a book on price theory (Untersuchungen über die Theorie des Preises) in 1889, that, as Schmidt (2004) has discovered contains a mathematical derivation of the envelope theorem and also some diagrammatic exercises with cost curves that beats Viner's 50 years later.

The derivation in Untersuchungen is followed in the paper by Schmidt (2004) who discovered the contribution by the two Austrians, but I will follow Samuelson's derivation in Foundations of Economic Analysis, which may seem marginally more general. Let

$$z = f(x_1, \ldots\ldots x_n, \alpha) \tag{1}$$

And assume that the function is twice continuously differentiable. The reader may think of (1) as a profit function. There are many ways to prove the envelope theorem, but to start with we stick to Auspitz and Lieben (1889) and Samuelson (1947).

Assume an interior maximum which means that the first order conditions can be written as

$$\frac{\partial z}{\partial x_i} = f_i(x_1 \ldots x_n, \alpha) = 0 \qquad \text{i=1…n} \tag{2}$$

The optimal value function can be written

$$z^* = f(x_1(\alpha), \ldots\ldots x_n(\alpha), \alpha) \tag{3}$$

Then

$$\frac{dz^*}{d\alpha} = \sum_{i=1}^{n} f_i \frac{\partial x_i}{\partial \alpha} + f_\alpha = 0 + f_\alpha = \frac{\partial z^*}{\partial \alpha} \tag{4}$$

The second equality follows from equation (2). Equation (4) tells us that the total change (the total derivative) of the optimal value function with respect to α equals what you would get if the $\mathbf{x}$ vector is kept constant (the partial derivative). The derivative of the parameter (vector) can be looked upon as a cost benefit rule.

The higher order change is obtained by totally differentiation of equation (4). One obtains

$$\frac{d^2z^*}{d\alpha^2}=\sum_{i=1}^{n} f_i \frac{\partial^2 x_i}{\partial\alpha^2}+\sum_{i=1}^{n}\frac{\partial x_i}{\partial\alpha}\frac{d(f_i)}{d\alpha}+\sum_{i=1}^{n} f_{i\alpha}\frac{\partial x_i}{\partial\alpha}+f_{\alpha\alpha}=\sum_{i=1}^{n} f_{i\alpha}\frac{\partial x_i}{\partial\alpha}+f_{\alpha\alpha} \tag{5}$$

This is exactly the formula derived by both Samuelson and, more interestingly, Auspitz and Lieben. The higher order change when the $\mathbf{x}$ vector is kept constant gives

$$\frac{\partial^2 z^*}{\partial\alpha^2}=f_{\alpha\alpha} \tag{6}$$

Hence[7],

$$\frac{d^2z^*}{d\alpha^2}-\frac{\partial^2 z^*}{\partial\alpha^2}=\sum_{i=1}^{n} f_{i\alpha}\frac{\partial x_i}{\partial\alpha}>0 \tag{7}$$

Loosely speaking this tells us that the envelope curve must be locally less concave than the unrestricted curve. Samuelson proof of the result in equation (7) is based on a strict semi-definiteness of the quadratic form under maximum. Auspitz and Lieben claim something similar.

We cannot criticize Hotelling, Viner and followers for not citing the two Germans, because they very likely did not know of "Untersuchungen". Auspitz and Lieben seem to be outlaws in relation to the Austrian School, and their book was written in German, which at the time was not standard knowledge in an Anglo-American tradition. However, Irving Fisher claims that he was strongly inspired by the content in Untersuchungen when he wrote his Mathematical Investigations (1892). in the Theory of Value and Prices (1928) Edgeworth mentions Untersuchungen and he even reviewed it for Nature 1889. He in particular notes the presence of envelope curves[8]. In other words, they were also 42 years ahead of Harrod, Schneider and Viner in this respect.

Hotelling's (1932) use of envelope properties is connected to a result by F.Y. Edgeworth (1925) called *Edgeworth's Taxation Paradox.* He produced an example of a monopolistic railway company supplying two classes of passenger services at different prices and, unhindered by government interference, setting ticket prices so that profit is maximized.

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[7] This is proved by Samuelson by using the quadractic form of the Hessian matrix.

[8] Niehans (1990) and Schmidt (2004)

When the railway company has to pay a tax on each first class ticket it may happen that both the first and the second class tickets are decreased at profit maximum. Hotelling generalizes this result by proving rigorously what mechanisms are involved, both under monopoly and perfect competition. For the case of perfect competition he shows how a marginal change in taxation results in a first and second order change, where the first order change disappears, since demands equal supplies in general equilibrium. The second order change consists of the so called Harberger triangles that were reinvented long after Hotelling's cost-benefit analysis of taxation.

Rene Roy's identity was produced in Roy (1947) and the proof of the result is in line with Auspitz and Lieben in that he uses the first order conditions of utility maximization. He also cites Irving Fisher as an example of an author of early mathematical economics. Fisher was, as mentioned above, inspired by Auspitz and Lieben, but he probably did not get stuck on the envelope side of their book.

Ronald Shephard's Lemma appears on page 13 in Shephard (1953) and follows from results from convex theory and by an old theorem by Minkowski (1911), but it is also derived from a distance function approach.

One cannot help to reflect over why so many economists, typically independent of each other, have ended up proving the same result over and over again, and getting credit in terms of their own name attached to the result. My reflections have so far not ended up in any complete answer, but the following story by Erwin Diewert explains how Shephards lemma surfaced[9]:

I *was a Ph.D student at Berkeley, 1964-1968 (got my degree in 1969) so I did indeed overlap with Shephard at that time but I did not take any courses from him. I did see him occasionally in the Econometrics Workshop, which I attended for the 4 years I was at Berkeley so I knew who he was.*

I had a summer job in Ottawa in 1967 for the Department of Manpower and Immigration, trying to predict the demand for different types of labour. I was not happy with the Leontief type production functions that they were estimating at the

[9] E.mail communication with Erwin Diewert.

time so I thought that I would generalize the functional form to allow for substitution. The demand function I estimated had the following functional form for input 1 say:

(1) $$x_1 = \{a_{11} + a_{12} p_2 p_1^{-1} + ... + a_{1n} p_n p_1^{-1}\} y$$

where

$x_1 =$ *demand for input 1;*

p_n = *nth input price*

$y = output$

I presented my empirical results on Manpower demand in Canada using the above functional form in the econometric workshop. Dan McFadden was in the audience and said to me: "Erwin, your demand functions are not integrable!" I had no idea what he was talking about but he told me to read his 1966 Berkeley working paper on duality theory as well as Shephard's 1953 book, which I did. And I realized that if I simply took the square roots of the input price ratios on the right hand side of the demand equations of the form (1), then my demand functions would be integrable (with symmetric conditions imposed) and thus was born the Generalized Leontief production and cost functions. In my reading of Shephard's 1953 book, I realized that he provided a proof of "Shephard's Lemma" starting from the cost function (as opposed to Hicks in Value and Capital, who started with the production or utility function and derived the result). So I named Shephard's result "Shephard's Lemma" in my first Berkeley discussion paper on the Generalized Leontief Production Function (later published in the Journal of Political Economy in 1971) and in my 1969 thesis. So I was certainly influenced by Shephard but at that stage, it was only by reading his book. I went on and did my thesis on flexible functional forms under the direction of McFadden.

Later on during the 1970s and 1980s, our paths crossed at the Index number workshops that Wolfgang Eichhorn held in Karlsruhe. At first Shephard did not much like me (he thought that I was stealing his stuff) but later on, he realized that my papers were making him more famous than ever and we got along quite well.

So that is my story on the origins of the term "Shephard's Lemma".

4. Calculus of Variations and Envelope Theorems

The calculus of variations was initiated by Galileo Galilei (1564-1642) and Johann Bernoulli (1667-1748). Galilei was thinking about the brachistochrone problem, *"the slide of quickest decent without friction*". He did not solve it himself. It was Johann Bernoulli that settled the problem in 1696. He showed that the optimal curve is a cycloid; a circle shaped curve that is mapped from a fixed point on the periphery of a circle when the circle rotates. A quarter of a century later Bernouili proposed to his student Leonard Euler to take up the task of finding general methods to solve similar problems. This started the calculus of variations. In 1759 Euler received a letter from the young Joseph Lagrange that contained a proof of necessary conditions which also involved the germ of the multiplier rule for a calculus of variations problem with constraints. Euler wrote back and told Lagrange that he also had done progress but would refrain from publishing his results until Lagrange had published his. That is scientific generosity!

To be honest I have not even skimmed the literature on the calculus of variations after Euler, but I doubt there is any envelope result until the dissertation by Ernst Zermerlo in 1894.It is, however , not easy to understand. I have tried to read Zermerlo's thesis, and it was by no means easy. However, as far as I can understand, he was up to finding necessary conditions for an optimal path. The envelope theorem comes as the closing key result of the thesis. The problem looks very much the same as what a general calculus of variations problem looks like today. He starts from Weierstrass[10] who was standing on the axis of Euler and Lagrange. The diagram below is an illustration of the theorem.

[10] Karl Weierstrass (1815-1897) German mathematician who did important contribution to real analysis and the calculus of variations. He introduced uniform convergence into mathematics. He also showed that there exists a closed graph that has no tangent at any point. A Brownian motion process is one example. I am not sure that Bachelier (1900) and Einstein (1905) discovered that.

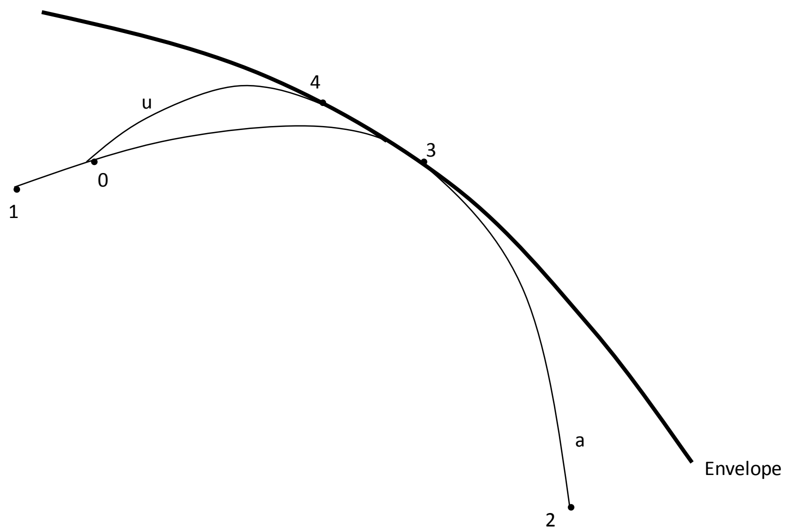

Figure 1: Illustration of Zermerlo's envelope theorem.

The bold curve is an envelope to the optimal solution curve a *and* u is a curve that starts at zero on the optimal curve and joins the envelope in point 4. The optimal curve starts at 1 and ends at 2, and at 3 it is a tangent to the envelope. The optimal value function is given by $J_{12} = \int_{t_1}^{t_2} F^*(\dot{y}(t), y(t), t; k)dt$. Zermerlo proves that the variation 043 from 03 vanishes when the Value functions are integrated in the following manner

$$J_{043} - J_{03} = \int_{\lambda_4}^{\lambda_3} E(\lambda)d\lambda = 0$$

This means that

$$J_{10432} = J_{12}$$

The disturbed part of the optimal path does not matter. The details are available in Zermerlo[11] (1894), but I do not recommend economists to spend too much time on them.

[11] Zermerlo was not the only one that produced envelope theorems in the calculus of variations. Darboux (1894) and Knerser (1898) were two others. Zermerlo is today quite well known among game theorists. He was the first to discuss whether chess has a solution in Zermerlo (1913). His theorem says that either white or black has a winning strategy or both can force a draw. The proof had some blemishes, pointed out by König (1927) and the proof was rectified by both of them. There are two paragraphs in König (1927) where Zermerlo's way to fix his proof is shown. See Larson (2008).

My guess is that the theorem is related to the same class of results as the Fundamental Theorem of the Calculus of Variations[12]. It is, however, not clear to me how Zermerlo's theorem can help to find the optimal path. He comments his accomplishment in the following manner (author's translation from German).

"This result is essentially a generalization of a property of a catenary first discovered by mr Lindelöf (Moigno and Lindelöf, Lecons di Calcul Differential e Integral IV Calcul de Variations) covering the contents of surfaces of revolution $\int yds$ by which two surfaces have separated tangents in terms of envelopes . On the other hand, it lacks me so far a simple criterion for the existence of a general envelope from the assumed properties."

The function $E(\lambda)$ is a construction of Weierstrass that is non negative but zero in this particular situation. A catenary is the curve that an idealized hanging chain or cable assumes when supported at its ends and acted only by its weight. A surface of revolution is a surface in Euclidian space created by rotating a curve around a straight line.

4. Optimal Control Theory

The envelope theorems in optimal control theory are in principle of the same character as the static ones. The "classical result" must, in a sense, have been known already by William Rowan Hamilton, who[13] in 1833 reformulated classical mechanics into Hamilton dynamics. He built on a previous reformulation of Joseph Lagrange from 1788. The Hamilton equations provide a new and equivalent method of looking at classical mechanics. They are not simpler to solve but provide new insights. I do not know physics, so I will give the economic interpretation of the Hamilton equations by starting from a Ramsey problem[14]. Ramsey's version was an optimal intertemporal saving problem that he solved in spite of the fact that the value function was unbounded[15]. The following optimization problem is, except for the upper integration level of the value function, a version of Ramsey's original problem.

[12] See e.g. Seierstad and Sydsaeter (1987) chapter 1.
[13] He is also well known for his four dimensional complex number theory (quarternions) and his drinking habits. He died from gaut 63 years old.
[14] Developed by Frank Plumpton Ramsey (1928)
[15] The reason was that he did not like discounting due to ethical reasons.

$$\underset{c(t)}{Max}\int_0^T f_0(\mathbf{x(}t),\mathbf{c}(t),t;\alpha)dt \qquad (8)$$

subject to

$$\dot{\mathbf{x}} = f(\mathbf{x(t),c(t)},t;\alpha) \qquad (9)$$

$$\mathbf{x}(0) = \mathbf{x}_0 \qquad (10)$$

$$\mathbf{x}(T)\ \ free \qquad (11)$$

Here, $\mathbf{x_0}$ is the value of the vector of stocks at the starting time, and the last condition in (11) means that there are no restrictions on the stocks at the time horizon. The vector $\mathbf{c(t)}$ is a consumption vector, *t is a time variable and* α *is a parameter (vector)*.

The first "envelope result" follows from Hamilton himself. From the maximum principle we can write the optimized Hamiltonian as

$$H^*(t) = f_0[\mathbf{x}^*(t),\mathbf{c}(x^*(t;\alpha),t;\alpha] + \boldsymbol{\lambda}(t;\alpha)f[\mathbf{x}^*(t;\alpha)c(\mathbf{x}^*(t;\alpha),t;\alpha] \qquad (12)$$

Where $\boldsymbol{\lambda}(t;\alpha)$ is a vector of adjoint variables. An economist would use "co-state variables" since capital stocks are state variables. We can rewrite (12), since "consumption is optimized" out, in the following manner

$$H^* = H^*(\mathbf{x}^*(t;\alpha),\boldsymbol{\lambda}(t;\alpha),t;\alpha) \qquad (13)$$

Assuming differentiability with respect to time yields

$$\frac{dH^*}{dt} = \frac{\partial H^*}{\partial \mathbf{c}}\dot{\mathbf{c}} + \frac{\partial H^*}{\partial \mathbf{x}}\dot{\mathbf{x}} + \frac{\partial H^*}{\partial \boldsymbol{\lambda}}\dot{\boldsymbol{\lambda}} + \frac{\partial H^*}{\partial t} \qquad (14)$$

Using (9) for $\frac{\partial H^*}{\partial \boldsymbol{\lambda}} = \dot{\mathbf{x}}$, the optimality condition for the co-state $\dot{\boldsymbol{\lambda}} = -\frac{\partial H}{\partial \mathbf{x}}$ and $\frac{\partial H^*}{d\mathbf{c}} = 0$ we obtain

$$\frac{dH^*}{dt} = \frac{\partial H^*}{\partial t} \qquad (15)$$

i.e. the total derivative of the Hamiltonian equals the partial derivative of the Hamiltonian

The value of the Hamiltonian in H-mechanics describes the total value of the energy of the system. For a closed system, equation (15) is the sum of the kinetic and potential energy in the system that are governed by the Hamiltonian equations

$$\dot{\mathbf{x}} = \frac{\partial H^*}{\partial \mathbf{x}} \qquad \dot{\boldsymbol{\lambda}} = -\frac{\partial H^*}{\partial \boldsymbol{\lambda}} \tag{16}$$

Where $\boldsymbol{\lambda}(t)$ are called generalized momenta, and $\mathbf{x}(t)$ are called generalized coordinates. If the system is conservative, the Hamiltonian will be constant over time ($\frac{dH}{dt} \equiv 0$). In economics we typically use discounting. Given that $\dot{x} = f(\bullet)$ is independent of *t* this means that

$$\frac{dH^*}{dt} = \frac{\partial H^*}{\partial t} = -\theta f_0(\bullet)e^{-\theta t} \tag{17}$$

This can be integrated to yield

$$H^*(t) = \theta \int_t^T f_0^*(s;\alpha)e^{-\theta s} ds + H^*(T) \tag{18}$$

For the typical case in a Ramsey world, $T \to \infty$ and $\lim_{T\to\infty} H(T) = 0$. This means that the optimal value function of the optimal control problem is proportional to the maximized Hamiltonian. The factor of proportionality is the discount rate θ. A now well-known result proved by Martin Weitzman in (1976). As we will show it also follows more directly from the Hamilton-Bellman-Jacobi equation (HJB).

5. The Maximum Principle and Cost Benefit analysis

Cost Benefit analysis is certainly an economic technique that has been improved by envelope results. The first time this was done is probably Hotelling's discussion of Edgeworth's taxation paradox, where he uses that excess demand in general equilibrium is zero implying that all the terms of first degree vanishes in the tax rates, to come up with his result.

Here we will show how cost benefit analysis is done in a dynamic context using envelope properties.

Let us start by rewriting the optimal value function above in the following manner[16]

$$
\begin{aligned}
&V(t,T,\mathbf{x}_t;\alpha) = \\
&\int_t^T \{ f_0[x^*(s,\alpha),\mathbf{c}^*(s,\alpha),s;\alpha]e^{-\theta s} + \boldsymbol{\lambda}(s,\alpha) f[\mathbf{x}^*(s,\alpha),\mathbf{c}^*(s,\alpha);\alpha] - \dot{\mathbf{x}}^*(s,\alpha)\} ds = \qquad (19) \\
&\int_t^T H^*[\mathbf{x}^*(s,\alpha),\mathbf{c}^*(s,\alpha),\boldsymbol{\lambda}(s,\alpha),s;\alpha]ds + \boldsymbol{\lambda}(t)\mathbf{x}(t) - \boldsymbol{\lambda}(T)\mathbf{x}(T) + \int_t^T \dot{\boldsymbol{\lambda}}(s)\mathbf{x}(s)ds
\end{aligned}
$$

To obtain the third line partial integration has been used. We can now differentiate the value function with respect to the lower integration level, the upper integration level and the capital stock at time *t*, $\mathbf{x}(t) = \mathbf{x}_t$.

We start with the derivative of the lower integration level to get

$$
\frac{\partial V}{\partial t} = -H^*(t) + \dot{\boldsymbol{\lambda}}(t)\mathbf{x}^*(t) + \boldsymbol{\lambda}(t)\dot{\mathbf{x}}(t) - \dot{\boldsymbol{\lambda}}(t)\mathbf{x}^*(t) = -H(t) \qquad (20)
$$

Since, $\mathbf{x}(t) = \mathbf{x}_t$ is a constant $\dot{\mathbf{x}}(t) = 0$. For similar reasons $\frac{\partial V}{\partial T} = H^*(T)$. Finally it follows

Immediately from equation (19) that $\frac{\partial V}{\partial \mathbf{x}_t} = \boldsymbol{\lambda}(t)$. The latter vector (the adjoint vector) tells us

about the value of an extra unit of capital at time *t* (the shadow prices of the capital stocks or state variables.

[16] This trick is due to an idea by Leonard (1987). He is also worth an envelope theorem.

What has the above to do with cost benefit analysis? One answer is that we can treat α as a vector of parameters and change this vector by adding increments $d\alpha = [d\alpha_1, ..., d\alpha_n]$ and add try to evaluate how this changes the optimal value function. The general idea would be to totally differentiate the vale function with respect to the parameters. Since the parameter vector is everywhere in the Hamiltonian this result in a mess. However, by adding the parameter vector as the state variable to the Hamiltonian by putting

$$\begin{aligned} &\dot{\alpha} = 0 \\ &\alpha(t) = \alpha \end{aligned} \tag{21}$$

with shadow price vector $\mu(s)$, we now from the maximum principle that

$$\dot{\mu}(s) = -\frac{\partial H^*(s)}{\partial \alpha} \tag{22}$$

Integrating forwards yields

$$\mu(T) = \mu(t) - \int_t^T \frac{\partial H^*(s)}{\partial \alpha} ds \tag{23}$$

Hence the value of the project is

$$\mu(t) = \mu(T) + \int_t^T \frac{\partial H^*(s)}{\partial \alpha} ds \tag{24}$$

Typically $\mu(T) = 0$

Hence,

$$\mu(t) = \int_t^T \frac{\partial H^*(s)}{\partial \alpha} ds \tag{25}$$

In other words, differentiation with respect to parameters and initial conditions give similar answers. The reason is that parameters can be upgraded to “stiff” state variables.

For an infinite time horizon problem with a finite project Li and Löfgren (2008) has shown that the present value sum of the direct perturbations of consumption and investment over the finite project period will give us the value of the project. Note that the cost-benefit rule both in equation (24) and the result in Li and Löfgren (2008) does not involve indirect general equilibrium effects. The reason is that we obtain envelope properties along the optimal path[17]. Li and Löfgren in addition show that the direct net effect during the project period is enough to obtain a correct answer. To see this we start from a rather straight forward cost-benefit rule and a related cost-benefit rule by Dixit et al. 1980.

The first two Propositions looks like this:

***Proposition 1**: The effect of a small policy reform, $d\alpha$, over the period $[t,T]$ measured on the optimal value function is the same as change as the present value of the future consumption over the period $[t,\infty)$*

$$dw_2(\alpha) = \int_t^{\infty} [\mathbf{p}^*(\mathbf{s})\mathbf{dc}(\mathbf{s}) e^{-\int_t^s r(\tau)d\tau}] ds$$

This rule is close to trivial but we will use the second cost-benefit rule to prove that. The second rule reads:

***Proposition 2(Dixit et al.):** The effect of a small policy reform over the period $[t,T]$ which results in changes in in consumption, investment and the stocks of capital through the changes $[\mathbf{\Delta c(s), \Delta I(s), \Delta k(s)}]_t^T$ is profitable if and only if*

$$\int_t^T [\mathbf{p}^*(\mathbf{s})\mathbf{\Delta c}(\mathbf{s}) + \mathbf{q}^*(\mathbf{s})\mathbf{\Delta I}(\mathbf{s}) + \boldsymbol{\kappa}^*(\mathbf{s})\mathbf{\Delta k}(\mathbf{s})] e^{-\int_t^s r(\tau)d\tau} ds > 0$$

Here $\mathbf{p}^*(\mathbf{s})$ is an optimal price vector, $\mathbf{q}^*(\mathbf{s})$ is the price vector of investment goods and $\boldsymbol{\kappa}^*(t)$ is the cost of holding capital. Note here that the project stops at time **T.** However, there are still indirect effects involved, which we are able to get rid of by looking at envelope properties in Proposition 2.

When this done we are able to write down a result that has to do with a change in NNP. It can be written:

***Proposition 3 (Li and Löfgren 2008):** A small project $\mathbf{d\alpha}$ in a market economy over the time interval $[t,T]$ leads to direct changes in consumption an investment or in vector form $\mathbf{d\tilde{c}} = \frac{\partial \delta}{\partial \alpha} d\alpha$, $\mathbf{d\tilde{I}} = \frac{\partial \mathbf{I}}{\partial \alpha} d\alpha$ .The value of the project is the changes in NNP*

The project is profitable if and only if

[17] The proofs are also available in Li and Löfgren (2008), where one can show that all results are equivalent.

$$\int_t^T [\mathbf{p}^*(s)\mathbf{d}\tilde{c}(s) + \mathbf{q}^*(s)\,\mathrm{d}\tilde{\mathrm{I}}(s]\mathrm{e}^{-\int_t^T r(\tau)d\tau} \; ds > 0$$

Note that $\mathbf{d\tilde{c}}$ and $\mathrm{d}\tilde{\mathrm{I}}(\mathrm{s})$ are direct changes of the project over *[t, T]* and the value of capital in Proposition 2 disappears. Moreover, all three result will give the same results, but Proposition 3 is easier to handle. The proofs and envelope results to get rid of indirect effects are available in the appendix.

6. Stochastic cost –benefit rules

Similar envelope properties are at work also in stochastic optimization. One can in fact say that much of the deterministic version of Pontryagin's maximum principle follows from the stochastic version of optimal control theory based on Ito calculus (the HJB equation).

Let $u[c(t)]$ be a smooth strictly concave instantaneous utility function, where *c(t)* denotes per capita consumption. The optimization problem is to find an optimal consumption policy. The stochastic Ramsey problem can be written

$$E_0\left\{\int_0^T u[c(\tau)]e^{-\rho\tau}d\tau\right\}; \tag{26a}$$

subject to

$$dk(t) = [f[k(t)] - c(t) - (n-\sigma^2)k(t)]dt - \sigma k(t)dB(t) \qquad k_0 = k_t \tag{26b}$$

$$c(t) \geq 0 \quad \forall t$$

E_0 denotes that mathematical expectations are taken conditional on the information available at time zero. The capital stock per capita is denoted $k(t)$ and $f(k(t))$ is the production function. Population growth is denoted n, and σ is the standard deviation of the Brownian motion process $B(t)$ that governs population growth.

T is the first exit time from the solvency set[18] $G = \{k_\tau(\omega); k_\tau > 0\}$, i.e. $T = \inf\{\tau > s; k_\tau(\omega) \notin G\} \leq \infty$. In other words, the process is stopped when the capital stock per capita becomes non-positive (when bankruptcy occurs). The stochastic differential equation above is not Geometric Brownian motion and we cannot guarantee that $k(\tau)$ stays non-negative, i.e. that bankruptcy does not occur[19].

Since there is no fundamental time dependence, only a discount factor with a constant utility discount rate, one can show that the optimal path is independent of the starting point. This means that we can prove that[20] $V(t,k_t) = V(0,k_t)e^{-\theta t}$ and the so called Hamilton-Jacobi –Bellman (HJB) equation can be written in the following manner

$$\theta W(t,k_t) = \underset{c}{Max}\left[u(c(t) + W_k h(k,c;\sigma^2,n) + \frac{1}{2}\sigma^2 k^2 W_{kk} \right] \quad (27)$$

where $W(k_t) = e^{\theta t}V(t,k_t) = V(0,k_t)$, $h(k,c;\sigma^2,n) = dk$ and θ is the discount rate. We can now define a co-state variable *p*(*t*) as

$$p(t) = W_k(k) \quad (28)$$

and its derivative

$$\frac{\partial p(t)}{\partial t} = W_{kk}(k) \quad (29)$$

We can now write

$$\theta W(k_t)) = u(c^*) + ph(k,c^*;\sigma^2,n) + \frac{1}{2}\frac{\partial p}{\partial k}\sigma^2 k^2 = H^{c^*}(k,p,\frac{\partial p}{\partial k}) \quad (30)$$

[18] *G* is simply the real positive line $[0,\infty)$

[19] A hard question is whether it occurs with probability one.

[20] A proof is available in Li and Löfgren (2009).

The function $H^{c^*}(\cdot)$ can be interpreted as a "generalized" optimized Hamiltonian in current value terms. Similar to Weitzman theorem ($H^* = \theta V^*$), the HJB equation shows that the generalized current value Hamiltonian is directly proportional to the optimal value function. Moreover, and also interesting, is that by putting $\sigma = 0$ equation (30) collapses to Weitzman's theorem. In fact, also the co-state and state equations collapses to those of the maximum principle[21]. One can say that most of the maximum principle follows as a special case from stochastic optimal control.

Moreover, the cost benefit rule that was derived above looks the same, when you take expectations of the stochastic co-state equation that represents the cost benefit project. More precisely, it can be written:

$$p_\alpha(t) = E_t\{\frac{\partial H^*(\bullet)}{d\alpha}\} \tag{31}$$

Again envelope properties are involved. The reader is referred to a memoranda by Aronsson, Löfgren and Nyström (2003) and Aronsson Löfgren and Backlund (2004) for technicalities. Chapter 9 in the latter reference and Malliaris and Brock (1982) tell us more in detail how the HJB-equation and the maximum principle fit together.

Conclusions

It is not easy to sum up the contents of the paper. My curiosity may have put me astray, and the paper reminds me of a small smörgåsbord, which at least contains herring, salmon, fish eggs, sausages, meatballs , ham, pate' and almond potatoes. It is obvious that it does not contain the comprehensive story of envelope theorems, but I have hopefully conveyed the message on the importance of them for economic analysis. Optimization helps to produce them. Another message is that they are easy to handle. As Eugene Silberberg (1974, 1978) very wittedly has pointed out, the calculations can be carried out at the "back of an envelope". Finally, they are old and have been discovered by many.

[21] See Malliaris and Brock (1982)

Appendix I: Help with proofs of the three CBA rules in section 5.

The Formulas for Proposition 1 and 2 give the same results. This is also true for Proposition 3, but it is more difficult to prove.

The second proposition is due to Dixit et al. (1980) and is very well known (see e.g. Arrow et.al.2003; Dasgupta 2001), but it involves indirect effects under the project period, which is also true for proposition one, where the period ends in infinity. The third proposition is relatively new and involves only the project as such. We start by writing the optimal value function in terms of a simple trick.

$$W_0^*(\alpha) = \int_0^\infty u(\mathbf{c}^*(t,\alpha),\alpha)e^{-\theta t}dt = \int_o^\infty \{u[\mathbf{c}^*(t.\alpha),\alpha] + \boldsymbol{\lambda}(t,\alpha)[\mathbf{I}^*[\mathbf{c}(t,\alpha),\mathbf{k}^*(t,\alpha),t,\alpha] - \dot{\mathbf{k}}^*(t,\alpha)]\}\,\mathrm{dt}$$

(1) To show what is the cost-benefit rule we write the utility function in the following manner $u[c(s,\alpha)+\delta(\alpha)]$ where $\delta(\alpha)=0$ before the project is initiated and $\alpha=\alpha_0$. One direct effect of the project is $\frac{\partial u[\mathbf{c}^*(s,\alpha),\alpha]}{\partial \alpha} = \frac{\partial u[\mathbf{c}^*(s,\alpha),\alpha]}{\partial c}\frac{\partial \delta(\alpha)}{\partial \alpha}$ and the second direct effect is $\boldsymbol{\lambda}(t,\alpha)\frac{\partial \mathbf{I}^*}{\partial \alpha}$

By total differentiating the optimal value function with respect to α we get

$$dW_0 = \int_0^\infty \{\frac{\partial u^*}{\partial \mathbf{c}}[\frac{\partial \mathbf{c}^*}{\partial \alpha}\frac{\partial \delta}{\partial \alpha}] + \boldsymbol{\lambda}(t,\alpha)[\frac{\partial \mathbf{I}^*}{\partial \mathbf{c}}\frac{\partial \mathbf{c}^*}{\partial \alpha} + \frac{\partial \mathbf{I}^*}{\partial \mathbf{k}}\frac{\partial \mathbf{k}^*}{\partial \alpha} + \frac{\partial \mathbf{I}^*}{\partial \alpha} - \frac{\partial \dot{\mathbf{k}}^*}{\partial \alpha}] + \frac{\partial \boldsymbol{\lambda}(t,\alpha)}{\partial \alpha}[\mathbf{I}^*(\bullet) - \dot{\mathbf{k}}^*(\bullet)]\}\,\mathrm{e}^{-\theta t}\,dt$$

(2)

The last term of the equation is equal to zero since $\mathbf{I}^*(\bullet) = \dot{\mathbf{k}}^*(\bullet)]$ and we simplify also by integrating the term $\boldsymbol{\lambda}\frac{\partial \dot{\mathbf{k}}^*}{\partial \alpha}$ and we get:

$$\int_0^\infty \boldsymbol{\lambda}(t,\alpha)\frac{\partial \dot{\mathbf{k}}^*}{\partial \alpha}e^{-\theta t}dt = [\boldsymbol{\lambda}(t,\alpha)\frac{\partial \mathbf{k}^*}{\partial \alpha}]_0^\infty - \int_0^\infty [\dot{\boldsymbol{\lambda}}^*(t,\alpha) - \theta \boldsymbol{\lambda}^*(t,\alpha)]\frac{\partial \mathbf{k}^*}{\partial \alpha}\,\mathrm{e}^{-\theta t}\,dt = -\int_0^\infty \mathbf{s}^*(t,\alpha)\frac{\partial \mathbf{k}^*}{\partial \alpha}e^{-\theta t}dt$$

(3)

Here $\mathbf{s}(t,\alpha)$ can be interpreted as the cost of keeping the capital an infinitesimal period of time dt. The other component is zero since $\mathbf{k}(0)$ is fixed and the tranversality condition goes to infinity as zero. We can now write the first equation in the following manner:

$$dW_0(\alpha) = \int_0^\infty \{\frac{\partial u^*}{\partial \mathbf{c}}[\frac{\partial \mathbf{c}^*}{\partial \alpha} + \frac{\partial \delta}{\partial \alpha}] + \boldsymbol{\lambda}(t,\alpha)[\frac{\partial \mathbf{I}^*}{\partial \mathbf{c}}\frac{\partial \mathbf{c}^*}{\partial \alpha} + \frac{\partial \mathbf{I}^*}{\partial \mathbf{k}}\frac{\partial \mathbf{k}^*}{\partial \alpha} + \frac{\partial \mathbf{I}^*}{\partial \alpha}] + \mathbf{s}^*(t,\alpha)\frac{\partial \mathbf{k}^*}{\partial \alpha}\}e^{-\theta t}dt \quad (4)$$

The expression in equation 4 can be further simplified. To start with we can reduce the project from an infinite planning horzion to the project interval. The reason for the necessary conditions for optimum $\frac{\partial u^*}{\partial c} + \boldsymbol{\lambda}(t,\alpha)\frac{\partial \mathbf{I}}{\partial c} = 0$ and $\dot{\boldsymbol{\lambda}} - \theta\boldsymbol{\lambda} = -\frac{\partial H}{\partial \mathbf{k}}$ along the path $\alpha=\alpha_0$, and the direct effects are in the project period [0, T].

By dividing the marginal utility with income in a utility metric μ we get the price vectors $\mathbf{p}^*(t) = \frac{\partial u^*}{\partial \mathbf{c}} / \mu(t)$. $\mathbf{q}^*(t) = \boldsymbol{\lambda}^*(t) / \mu(t)$ och $\boldsymbol{\kappa}^*(t) = \mathbf{s}^*(t) / \mu(t)$ for consumption, investment and capital and we can write:

$$dW_0(\alpha) = \int_0^T [\mathbf{p}^*(t)\Delta c + \mathbf{q}^*(t)\Delta\mathbf{I}(t) + \boldsymbol{\kappa}^*(t)\Delta\mathbf{k}]\mu(t)e^{-\theta t}dt$$

where $\Delta\mathbf{c} = (\frac{\partial \mathbf{c}}{\partial \alpha} + \frac{\partial \delta}{\partial \alpha})d\alpha, \mathbf{\Delta I} = (\frac{\partial \mathbf{I}^*}{\partial \mathbf{c}}\frac{\partial \mathbf{c}^*}{\partial \alpha} + \frac{\partial \mathbf{I}^*}{\partial \mathbf{k}}\frac{\partial \mathbf{k}}{\partial \alpha})d\alpha$ and $\Delta\mathbf{k} = \frac{\partial \mathbf{k}^*}{\partial \alpha}d\alpha$

Since $\dot{\mu} = \mu(t)[r(t) - \theta]$, $\mu(0) = \mu_0$ we can solve the differential equation to get

$\mu(t)e^{-\theta t} = \mu(0)e^{-\int_0^t r(\tau)d\tau}$ and use the left hand side of the equation and put $\mu(0) = \mu_0 = 1$. This proves Proposition 2, i.e.

$$dW_0(\alpha) = \int_0^T [\mathbf{p}^*(t)\Delta\mathbf{c} + \mathbf{q}^*(t)\Delta\mathbf{I}(t) + \boldsymbol{\kappa}^*(t)\Delta\mathbf{k}]e^{-\int_0^t r(\tau)d\tau}dt$$

We can now also show that the two last terms in the integrand above is an exact differential such that:

$$\int_0^T [\mathbf{q}^*(t)\Delta\mathbf{I}(t) + \boldsymbol{\kappa}^*(t)\Delta\mathbf{k}(t)]e^{-\int_0^t r(\tau)d\tau}dt = \mathbf{q}^*(T)\Delta\mathbf{k}(T)[\mathrm{e}^{-\int_0^T r(\tau)d\tau} \quad (5)$$

Note that $\Delta\mathbf{k}(0)$ is fixed! Equation (5) shows how the future looks for consumption during the interval from $[\mathrm{T}\,\infty)$, i.e.

$$\int_T^\infty \mathbf{p}^*(t)\,\mathrm{d}\,\mathbf{c}(t)e^{-\int_t^\infty r(\tau)d\tau d}dt = \mathbf{q}^*(T)\Delta\mathbf{k}(T)[\mathrm{e}^{-\int_0^T r(\tau)d\tau}$$

This follows because we can split Proposition 1 and Proposition 2. From Proposition 2 we can use an envelope property to understand proposition 3 to unfold Proposition 2 into Proposition 3. To be clear we use equation 4 to get:

$$[\frac{\partial u^*}{\partial \mathbf{c}} + \boldsymbol{\lambda}(t,\alpha)\frac{\partial \mathbf{I}^*}{\partial \mathbf{c}}]\frac{\partial \mathbf{c}^*}{\partial \alpha} = 0 \text{ och } \mathbf{s}^*(t,\alpha)\frac{\partial \mathbf{k}^*}{\partial \alpha} + \lambda(\mathrm{t},\alpha)\frac{\partial \mathbf{I}^*}{\partial \mathbf{k}}\frac{\partial \mathbf{k}^*}{\partial \alpha} = -[\frac{\partial H^*}{\partial \mathbf{k}} - \frac{\partial H^*}{\partial \mathbf{k}}]\frac{\partial \mathbf{k}^*}{\partial \alpha} = 0$$

We has now proved Proposition 3 in the main text. We write $d\widehat{\mathbf{c}}(t) = \frac{\partial \delta(t,\alpha)}{\partial \alpha}d\alpha,\ d\widehat{\mathbf{I}}(t) = \frac{\partial \mathbf{I}}{\partial \alpha}d\alpha$ and the proposition reads:

$$\int_t^T [\mathbf{p}^*(s)\mathbf{d}\widehat{c}(s)+\mathbf{q}^*(s)\,\mathrm{d}\widehat{\mathrm{I}}(s]\,\mathrm{e}^{-\int_t^T r(\tau)d\tau}\,ds$$

Appendix II: Help with the stochastic differential equation in section 6 for the capital per capita using Ito's Lemma

Many clever mathematicians have had some problems with the derivation below. Let us start from the stochastic differential equation:

$$dL = nL(t)dt + \sigma L(t)dB(t) \qquad (1)$$

We can now transform the uncertainty of growth in labor to an uncertainty into the growth in the capital per capita $k(t) = K(t)/\mathrm{L}(t)$. We start by defining $k(t) = Z(t, \mathrm{L(t)})$ and use Ito's Lemma. We get:

$$\begin{aligned}
dk &= \frac{\partial Z}{\partial t}dt + \frac{\partial Z}{\partial L}dL + \frac{1}{2}[\frac{\partial^2 Z}{\partial K^2}dK^2 + 2\frac{\partial^2 Z}{\partial K\partial L}dKdL + \frac{\partial^2 Z}{\partial L^2}dL^2] = \\
&= \frac{1}{L}[Lf(k) - C]dt + \frac{-K}{L^2}(nLdt + \sigma LdB) + \frac{1}{2}[\frac{2}{L^3}K\sigma^2 L^2 dt] = \\
&= [f(k(t)) - c(t) - (n-\sigma^2)k(t)]dt - k(t)\sigma dB(s)
\end{aligned} \qquad (2)$$

where $c(t)$ is consumption per capita. Note that the differentials dK^2 and $dKdL$ disappears since they remain second order terms at the end of the second line, while ' $dL^2 = \sigma^2 L^2 dt$, and the last term at the end of the last line stands for the stochastic part. Note also that $Lf(k) = Lf(k,1) = F(K,L)$, since the production function is homogeneous of degree one.

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