The Minimum Wage and a Non-Competitive Market for Qualifications

Gauthier Lanot
Department of Economics,
Umeå School of Business and Economics,
Umeå University,
901 87 Umeå, Sweden
gauthier.lanot@umu.se

Panos Sousounis
Keele Management School
Keele University
Keele, ST5 5BG, UK
p.sousounis@keele.ac.uk

Abstract:
In this paper we consider an equilibrium model of demand and supply for several qualifications first in a competitive setting and then in a non-competitive setting. We propose a tractable analytical framework, i.e. when workers choose between qualifications according to a multinomial logit model of choice and when a CES production function describes the substitutions possibilities between the different types of labour. While in the competitive case the effects of the minimum wage are those we expect, in the imperfectly competitive case we find that a minimum wage can create unemployment and we find that the welfare of the population as a function of the minimum wage is not unimodal. We show furthermore that allowing one qualification to be exempted from the minimum wage does not mean that its relative demand is unaffected by changes to the minimum wage.

Keywords: Minimum wage, wage differentials, segmented labour markets, monopsony.

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Introduction

The effects of a minimum wage on employment and wages is often studied within the context a single labour market involving homogenous workers facing a variety of market conditions which invalidate the competitive predictions: the most famous example arises in the context of monopsony (see Manning, 2003, for an analysis of its many forms). In such conditions a minimum wage will redistribute income to the least paid. The key policy question concerns the cost of the minimum wage in terms of market efficiency. When the labour market is dominated by a single monopsony firm a minimum wage that is less than the competitive wage will improve efficiency: it will reduce unemployment and it will increase the wage level. This is the textbook analysis which argues for the beneficial role of the minimum wage in the presence of market power on the labour market. By extension one may believe that any market where the demand side has "market power" over the supply labour will benefit from the introduction of a minimum wage.

In this paper we study the effect of a minimum wage in the context of a segmented labour market. The labour market is segmented since workers must choose among several qualifications which have distinct marginal productivities. We assume that workers decide on their preferred qualification by comparing the wages that each qualification commands. The total output in the economy depends on the particular mix of qualifications and qualifications can be substitutes or complements. We introduce some imperfections on this segmented labour market by allowing the aggregate production sector to determine the relative wages.

The objective of the paper is to describe a simple analytical framework within which we can analyse the effect of the minimum wage on the distribution of qualifications among the employed workers and in the population, and on the distribution of the equilibrium wages across qualifications. We do not account here for the important role of information asymmetry, search frictions or strategic interactions on the labour market. We show nonetheless that the case for the minimum wage may not be as straightforward as expected despite in our apparently elementary framework.

Grossman (1983) proposes a related analysis of the firm demand for labour for two distinct skill groups where the firm takes the wage of unskilled workers as given and determines the wage of skilled workers. Unskilled works is assumed to be subjected to the minimum wage and the effort of skilled workers responds both to their real wage and to their wage relative to the minimum wage. The model is therefore a model of efficiency wages on the market for skilled work. The analysis suggests that as the minimum wage increases the wage the firm pays for skill work increases because of the firm substitutes away from unskilled work and because the firm has an incentive to maintain the relative wage of skilled workers. Our modeling maintains the assumption that the proportion of workers supplied to the distinct qualification groups depends on the wages of each qualification relative to the competitive qualification.
In a related 2-sector framework, Lee & Saez (2012) analyse the contribution of the minimum wage in an optimal taxation schedule in a model of competitive labour markets. They assume that workers in the low paid sector are heterogenous in their productivity. An increase in the minimum wage is then welfare improving because it is able increases the average productivity of the low paid in work, while the least productive workers become unemployed first. They show that the optimal tax schedule takes advantage of this effect of the minimum wage. Because our focus is not on optimal taxation this is not the transmission mechanism we follow here.

Although the structure of the model is simple, the role of the different components of the model give a rich enough structure so that the effect of the minimum wage on the distribution of wages between qualifications depend on the model parameter values in a non-trivial way and possibly unexpected fashion given the model simple structure.

We show that the kind of market power we consider does not create unemployment but misallocation of workers to qualifications relative to the competitive outcome. A large enough minimum wage however can create unemployment and will not be able to correct completely for the efficiency loss due to the presence of market power. The intuition is clear: the minimum wage distorts relative wages which in turn distort the choices between qualifications and the demand for employment. Furthermore we show that the effect of the minimum wage on the population welfare is not necessarily monotone: welfare as a function of the minimum wage may have multiple maxima. This in turns imply that the “optimal” minimum wage may involve an optimal level of unemployment. Finally we show that the demand for workers with a qualification that is exempted from the minimum depends on the minimum wage.

Our model can capture some of the well documented heterogeneity observed between the low paid sectors in the UK for example. The reports of the UK Low Pay Commission (Low Pay Commission, 2013) describe the bite of the MW in several low pay occupations and show that the bite varies between occupations and that this pattern persists over time¹. In their review of the US evidence both Flinn (2010) and Neumark & Wascher (2008) similarly report that the bite of the minimum wage differs between low paid occupations.

The paper is organised as follows, we first describe the analytical framework in a competitive environment without a minimum wage, and then discuss the effect of the introduction of a minimum wage. In a third section, we extend the analysis to a model where the production sector acts in a monopsonistic manner on the market for qualifications and then show the effect of the

¹ The 2013 reports that 4 to 6% of men employed are covered by the UK national MW, for women this proportion is 6 to 9%. Moreover 19% of unqualified workers are paid at or below the NMW. For all workers 22 and over, the bite of the minimum wage (the ratio of the minimum wage to the median wage) is greater than 45% in 1999 and rising slowly to about 52% over the period to 2012. For the low paid sectors and over the same period the bite ranges from 60% to 80%, whereas for unqualified workers, the bite ranges between 80 and 90% over the period 2007/2012.
minimum wage in this context. Finally we discuss the effect of exemptions from the minimum wage in this framework and we conclude.

An Analytically Convenient Model of the Markets for Qualifications

We consider a large population of individuals who choose between \( Q+1 \) different qualifications. We normalise the population size to 1. We think of these individuals as prospective new entrants on the labour market. Their aim is to decide on the qualification which is best suited to their tastes given the wage that each qualification commands. We then model the demand side in the aggregate to determine endogenously the equilibrium wages which each qualification will command given all the parameters of the model. This captures in a reduced form manner the many interactions and frictions on the demand side which determine the overall technique. For example it may reflect the technological constraints of the production of qualification, or it may reflect the difficulty of retraining once a qualification has been acquired. These are important further issues which we do not consider here.

We assume that the choices between qualifications are such that it is the distribution of relative wages which determines the distribution of qualification choices. In particular we select a Multinomial Logit model of choice which simplifies the analysis (for a more general discussion of the logit model and its use in economic theory see Anderson, De Palma, & Thisse, 1992). We assume that individual \( i \)’s “utility” for qualification \( q \) is

\[
I_{q,i} = \delta_q + \delta \ln w_q + \eta_{q,i}, \quad q = 0, \ldots, Q, \tag{1}
\]

where \( \delta_q, \delta_q < +\infty \), is a qualification specific shifter (including for example the costs of obtaining the qualification evaluated in utility terms) and we set \( \delta_0 = 0 \), i.e. the \( 0^{th} \) qualification is set to be the reference qualification. \( \delta \), with \( \delta \geq 0 \), measures the marginal utility to changes in the qualification specific wage, and \( \eta_{q,i} \) is a unobserved individual and qualification specific random component which follows a Type 1 extreme value distribution.

The proportion of the population which chooses qualification \( q \) can be evaluated simply as:

\[
\pi_q^* = \frac{\exp(\delta_q + \delta \ln w_{q})}{\sum_{p=0}^{Q} \exp(\delta_p + \delta \ln w_{p})} = \frac{\exp\left(\delta_q + \delta \ln \frac{w_q}{w_0}\right)}{1 + \sum_{p=0}^{Q} \exp\left(\delta_p + \delta \ln \frac{w_p}{w_0}\right)}. \tag{2}
\]

The model of choice between qualifications is related to a Roy model, which it becomes if \( \delta = +\infty \). In that extreme case, workers choose the qualification which yields the largest wage with probability 1. In our context, this would lead all workers to make the same choice. Teulings (2003) analyses a more general framework where workers with distinct skills make different choices and are
assigned to a large number of alternative jobs according to a Roy model of choice in a competitive general equilibrium context.

Our analysis here is limited to a discrete number of labour markets and the assignment problem the workers solve in our case is simpler. Workers are heterogeneous and decide which qualification to acquire; furthermore we specify only one aggregate production technology where distinct qualifications have distinct marginal productivities. The individual unobserved component do not determine individual productivity, and therefore all workers with a given qualification command the same wage.

On the production side, we assume that the aggregate technology which determines the demand for labour inputs with alternative qualification is characterised by an increasing and concave function, \( \psi(\cdot) \), of a labour aggregate, \( F(\cdot) \):

\[
Y = \psi\left(F\left(N_0,\ldots,N_Q\right)\right),
\]

where \( N_q \) measures the quantity of input labour (for a particular generation) with qualification \( q \). The labour aggregate is of the CES form:

\[
F\left(N_0,\ldots,N_Q\right) = \left[N_0^\rho + \sum_{q=1}^{Q} \alpha_q N_q^\rho\right]^\frac{1}{\rho},
\]

with \( -\infty \leq \rho \leq 1 \) such that \( \rho = 1 - 1/\sigma \), where \( \sigma \), with \( \sigma \geq 0 \), measures the elasticity of substitution between labour types. Assuming that either aggregate profits are maximised or costs are minimised given the technology, the first order conditions for the demand of labour of type \( q \) relative to the demand for the \( 0^\text{th} \) qualification are

\[
\left[\frac{N_q}{N_0}\right]^{\rho-1} = \frac{w_q}{w_0 \alpha_q} \iff \frac{\pi_q}{\pi_0} = \alpha_q^{\sigma} \left(\frac{w_{q,0}}{w_0}\right)^{-\sigma},
\]

where \( \pi_q^D, q = 0,\ldots,Q \), stands for the demand for a proportion of workers with qualification \( q \), \( w_{q,0} \equiv w_q \) are the wages relative to the \( 0^\text{th} \) qualification. This particular functional form implies that the proportions on the demand side are of the multinomial logit type:

\[
\pi_q^D = \frac{\exp\left(\sigma \ln \alpha_q - \sigma \ln \left(w_{q,0}\right)\right)}{1 + \sum_{p=1}^{Q} \exp\left(\sigma \ln \alpha_p - \sigma \ln \left(w_{p,0}\right)\right)}, \text{ for all } q = 1,\ldots,Q,
\]

and

\[
\pi_0^D = \frac{1}{1 + \sum_{p=1}^{Q} \exp\left(\sigma \ln \alpha_p - \sigma \ln \left(w_{p,0}\right)\right)}.
\]
The equilibrium between the demand and the supply of the various qualification will determine the equilibrium relative wages \( w^*_{q,0} \equiv \frac{w^*_{q}}{w_0} \), \( q = 1, \ldots, Q \), such that:
\[
\pi^D_q (w^*_{q,0}) = \pi^S_q (w^*_{q,0}), \quad \text{for all } q = 1, \ldots, Q.
\]

In what follows we make the additional assumption that \( \delta + \sigma > 0 \) so that the equilibrium relative wages are such that
\[
\delta_q + \delta \ln w^*_{q,0} = \sigma \ln (\alpha_q) - \sigma \ln w^*_{q,0}, \quad \text{for all } q = 1, \ldots, Q,
\]
which implies that
\[
\ln w^*_{q,0} = \frac{\sigma \ln (\alpha_q) - \delta_q}{\delta + \sigma} \quad \text{for all } q = 1, \ldots, Q. \tag{7}
\]

We assume that qualification 0 commands such a high wage that \( \ln w^*_{q,0} \) is negative for all \( q \), i.e. \( w^*_{q,0} \leq 1 \) for all \( q \). Furthermore we will assume throughout that the list of \( \ln w^*_{q,0} \), the log of the equilibrium wages relative to qualification 0, is sorted in ascending order, we have
\[
\ln w^*_{1,0} \leq \ln w^*_{2,0} \leq \ldots \leq \ln w^*_{Q,0} \leq 0. \tag{8}
\]
Since \( \delta + \sigma \) is positive, this amounts to a restriction on the sign and the relative sizes of the differences \( \sigma \ln (\alpha_q) - \delta_q \). Note that the distribution of the quantities \( \sigma \ln (\alpha_q) - \delta_q \) determines the distribution of relative equilibrium wages and therefore determines the distribution of equilibrium wages. Unsurprisingly, the model predicts that the characteristics of the distribution of wages across qualifications depend on both the demand and supply parameters. Let \( \mu_q = \delta \ln (\alpha_q) + \delta_q \) and \( \nu_q = \sigma \ln (\alpha_q) - \delta_q \), so that we can express the equilibrium probabilities and relative wages as
\[
\pi^*_q = \frac{\exp \left( \frac{\sigma}{\delta + \sigma} \mu_q \right)}{1 + \sum_{p=1}^{Q} \exp \left( \frac{\sigma}{\delta + \sigma} \mu_p \right)}, \quad \text{for all } q = 0, \ldots, Q, \tag{9}
\]
and
\[
\ln w^*_{q,0} = \frac{\nu_q}{\delta + \sigma}, \quad \text{for all } q = 1, \ldots, Q, \tag{10}
\]
with \( \mu_0 = \nu_0 = 0 \). Condition (8) implies that \( \nu_1 \leq \nu_2 \leq \ldots \leq \nu_Q \leq 0 \) but does not imply a similar order between the quantities \( \mu_q \) (these quantities are each increasing with \( \nu_q \) but this does not determine their ordering).
However assuming that the number of workers with low paid qualifications, i.e. such that \( q \geq 1 \), are a small proportion of the workforce with the 0th
qualification in the competitive equilibrium, that is such that \( \pi_{q,0}^* < 1, \forall q = 1,\ldots,Q \), implies that \( \mu_q \leq 0, \forall q = 1,\ldots,Q \), since

\[
\ln \pi_{q,0}^* \equiv \ln \frac{\pi_q^*}{\pi_0} = \frac{\sigma}{\delta + \sigma} \mu_q, \text{ for all } q = 1,\ldots,Q.
\] (11)

Assuming \( \mu_q \leq 0, \forall q = 1,\ldots,Q \) it is then easy to conclude that for all \( q = 1,\ldots,Q \):

\[
\frac{\partial \ln \pi_{q,0}^*}{\partial \ln \sigma} = \frac{\ln \pi_{q,0}^*}{\delta + \sigma} \leq 0, \quad \frac{\partial \ln \pi_{q,0}^*}{\partial \ln \sigma} = \frac{\delta}{\delta + \sigma} \leq 0,
\] (12)

\[
\frac{\partial \ln w_{q,0}^*}{\partial \ln \delta} = -\frac{\ln w_{q,0}^*}{\delta + \sigma} \geq 0, \quad \frac{\partial \ln w_{q,0}^*}{\partial \ln \delta} = \frac{\sigma}{\delta + \sigma} \leq 0,
\] (13)

The competitive quantities decrease with the (technical) elasticity of substitution \( \sigma \). In an economy where the competitive wages and quantities are small relative to the 0\textsuperscript{th} qualification, a technology change which increases the substitutability between qualifications (as measured by a higher value of \( \sigma \)), all other things equal, leads to smaller competitive relative wages and relative quantities for qualifications \( q=1,\ldots,Q \). Furthermore, the relative wage bill for any qualification \( q=1,\ldots,Q \) must decrease since

\[
\frac{\partial \ln \pi_{q,0}^* w_{q,0}^*}{\partial \ln \sigma} = \ln \pi_{q,0}^* \cdot \frac{1 + \delta}{\delta + \sigma} \leq 0.
\]

Considering the supply side, a larger value of \( \delta \), all other things equal, leads to a larger sensitivity of qualification choice to relative wages, larger competitive relative wages while the competitive relative quantities are smaller. Overall the effect on the relative wage bill depends on \( \sigma \), since:

\[
\frac{\partial \ln \pi_{q,0}^* w_{q,0}^*}{\partial \ln \delta} = \ln w_{q,0}^* \cdot \frac{\sigma - 1}{\delta + \sigma},
\]

if \( \sigma < 1 \) the effect on the wage bill is positive, and negative otherwise.

We note further that the data of distribution of the competitive relative wages and quantities, \( (w_{1,0}^*,\ldots,w_{Q,0}^*,\pi_{1,0}^*,\ldots,\pi_{Q,0}^*) \) would not determine uniquely all the parameters of the model. There are several values for the parameters \( (\alpha_1,\ldots,\alpha_Q,\delta_1,\ldots,\delta_Q,\sigma,\delta) \) which would lead to the same values for the relative competitive wages and quantities \( (w_{1,0}^*,\ldots,w_{Q,0}^*,\pi_{1,0}^*,\ldots,\pi_{Q,0}^*) \).

In the competitive equilibrium the population is employed in its entirety: there is no unemployment caused by a mismatch between the supply and the demand for qualifications. Any possible disequilibrium is resolved by the price mechanism. The equilibrium we describe here leaves the equilibrium wage of the 0\textsuperscript{th} qualification unspecified, without any consequence for the distribution of the population across qualifications.
The Effect of a Minimum Wage in Competitive Markets

When a minimum wage is introduced, several effects come into play. First because the minimum involves an increase in the marginal cost of production the scale of production may vary, this is the usual scale effect. Second the minimum wage will affect some qualifications and not others. This will generate a redistribution of the demand and the supply of qualifications because the relative rewards and costs are modified. The new equilibrium set of wages must accounts for both effects.

In the presence of a minimum wage the calculus of the equilibrium wages and proportions is modified significantly since equating the proportions demanded and supplied of each qualification does not account for the possibility of some unemployment created by a mismatch between the demand side and the supply side.

Instead, the market equilibrium arises when the quantity of labour supplied with each qualification matches the quantity demanded. For any qualification where the minimum wage bites the supply side is determined relative to the wage of the $0^\text{th}$ qualification, while the demand side is determined by the same relative wage in equilibrium and the scale of production overall. In our simple model where the demand and supplies depend only on the wages relative to the wage paid for the $0^\text{th}$ qualification, all the analysis on the equilibrium for qualifications 1 to Q can be carried out in terms of the minimum wage relative to the wage for the $0^\text{th}$ qualification, say $\omega \equiv \frac{w}{w_0}$, where $w_0^{**}$ is the wage of qualification 0 when the minimum wage $w$ is in force.

The set of equilibrium relative wages in the labour markets where the minimum wage $w$ is imposed, $w^*_q = (w^*_0, \ldots, w^*_q, \ldots, w^*_Q)$, is such that

$$\pi^S_q (w^*) \geq \pi^D_q (w^*) \quad \text{for all } q=0\ldots Q. \quad (14)$$

and such that

$$\pi^S_q (w^*) = \pi^D_q (w^*) \quad (15)$$

for all qualification $q$ where the minimum wage does not bite, i.e. such that $w^*_q > \omega$.

Whenever $\omega > w^*_q$, for any of the $r$ least paid occupations $q = 1, \ldots, r$, $r \leq Q$, the relative supply of workers at the relative minimum wage $\omega$ will be larger than the demand at that wage. This is a consequence of our earlier assumption, relative to the competitive wage these qualifications are more expensive hence the demand for their services fall, as described by (4), whereas the relative supply increases, as described by (2).
Workers with qualifications which are not covered by the minimum wage are paid at the competitive relative wages $w_{q,0}^* > \omega$ such that relative demand and supply meet, as described by (7) for any $q > r$. In our model, the introduction of the minimum wage therefore affects directly the pay of the lowest paid workers (if they are employed), creates unemployment among these workers and leave the relative pay conditions for all workers uncovered by the minimum wage relatively unaffected (i.e. their pay differential relative to the $0^{th}$ qualification remains at the competitive level, the equilibrium quantities will change however). Introducing more labour markets and allowing for the endogenous distribution of the workers between sectors as we do here, does not change the textbook analysis of the effect of the minimum wage on a competitive labour market: the minimum wage creates some unemployment (we provide a complete analysis in the online appendix).

**Monopsony Power and the Labour Markets for Qualifications**

In this section, we describe the equilibrium outcomes when the demand side (the aggregate production sector) recognizes and incorporates into its decision problem the effects prices have on the distribution of workers across qualifications. Hence following the textbook’s intuition we would expect the minimum wage to be a relevant policy tool to improve welfare.

In general, it is possible for the demand side to decide on leaving some workers unemployed (as would be the case in the textbook exposition of the monopsony model), not because of any mismatch between demand and supply but simply because of the exercise of market power. However in the current model this is not a feasible equilibrium outcome. In equilibrium, any non-productive qualification, i.e. such that $\alpha_q = 0$, will be rewarded with a wage relative to the $0^{th}$ qualification equal to $0$ and therefore will neither be supplied nor demanded.

The analysis follows the “textbook” treatment of a monopsony, although in the current context the production side is involved in the management of the supply of labour across several sectors (our qualifications). An implicit assumption of the model, is that workers cannot move between qualifications, i.e. a worker with qualification $q$ enters the market for that qualification only and is not able to move to another close substitute qualification or even retrain. This is clearly a “real” phenomenon but beyond the scope of the current paper.

The model assumes that the aggregate production side does not determine the wage paid for workers with the $0^{th}$ qualification, indeed “monopsonistic” profits would be maximised w.r.t. $w_0$ with $w_0 \to 0$, i.e. the production side would understand that the supply of workers with the $0^{th}$ qualification is infinitely inelastic and therefore if the production side was to exploit its market power it would pay wages as close to $0$ as possible for all qualifications (or even would require the workers to pay for the right to work). This is a consequence of our early modelling assumptions which state that the supply for each qualification
depends only on the relative wages. To depart from such an extreme form of market power we assume instead that the production side determines its scale competitively, i.e. it determines the level of employment $\bar{N}_0$ for the workers with the $0^{th}$ qualification or, equivalently, it determines total employment $\bar{N}$, so as to maximise profit taking the wage of the highest paid qualification as given. Hence we consider an intermediate form of market power, where the production sector controls relative rewards and determines the employment of the highest paid worker competitively but does not determine absolute rewards (which would drive wages down as close to 0 as possible given our assumptions on qualification choice).

Instead of taking prices as given on the low paid qualifications, the production “understands” the relationship between the wages relative to the wage of the $0^{th}$ qualification and the distribution of the population between qualifications, that is for a given vector of relative wages $w_{0,0}, w_{q,0} < 1$, the firms expects the population to be distributed between qualification according to the probabilities:

$$\pi^*_q = \frac{\exp(\delta_q + \delta \ln w_{q,0})}{1 + \sum_{p=1}^Q \exp(\delta_p + \delta \ln w_{p,0})}, \text{ for } q = 0,..,Q.$$  

If $\delta = 0$ the proportions of workers with the different qualifications are fixed and independent of the relative wages, while if $\delta \to +\infty$, the qualification with the largest wage is supplied by the whole population and all the other qualifications are not supplied.

Assuming $0 < \delta < +\infty$, we can inverse these expression to find a relationship between the relative wages and the ratios $\pi^*_q, \pi^*_0 = \pi^*_q / \pi^*_0$, we find:

$$\ln \pi^*_q = \delta_q + \delta \ln w_{q,0}, \text{ for } q = 1,..,Q,$$

or equivalently, we consider the “inverse” supply curves:

$$w^*_{q,0} = \phi_q \pi^*_0, \text{ for } q = 1,..,Q,$$

with $\phi_q \equiv \exp\left(-\frac{\delta_q}{\delta}\right)$ for all $q > 1$.

For a given employment plan $N = (N_0, N_1,..,N_Q)$, and allowing the monopsony to account for the relative supplies, the total wage bill becomes:

$$\sum_{p=0}^Q w_p(N)N_p = w_0N_0 \left(1 + \sum_{p=1}^Q \phi_p \left(\pi^*_p\right)^{\gamma}\right),$$

with $\gamma \equiv 1 + \frac{1}{\delta}$.

The production side maximises its joint profits
\[
p_{\psi}(N_0 F(1, \pi_{1,0}, \pi_{2,0}, \ldots, \pi_{Q,0})) - w_0 N_0 \left(1 + \sum_{p=1}^{Q} \phi_p \left(\pi_{p,0}\right)^\gamma\right),
\]
with respect to \(\pi_{1,0}, \ldots, \pi_{Q,0}\) and \(N_0\).

Define \(L = N_0 F(1, \pi_{1,0}, \pi_{2,0}, \ldots, \pi_{Q,0})\) to measure the aggregate labour input and let
\[
\tilde{C}(\pi_{1,0}, \pi_{2,0}, \ldots, \pi_{Q,0}) \equiv \frac{1 + \sum_{p=1}^{Q} \phi_p \left(\pi_{p,0}\right)^\gamma}{F(1, \pi_{1,0}, \pi_{2,0}, \ldots, \pi_{Q,0})}
\]
measure the relative costs of employing \(\pi_{1,0}, \ldots, \pi_{Q,0}\) in each qualification \(q=1, \ldots, Q\) relative to the employment of workers with the \(0^{th}\) qualification. We observe that
\[
\max_{\pi_{1,0}, \pi_{2,0}, \ldots, \pi_{Q,0}} p_{\psi}(L) - L w_0 \tilde{C}(\pi_{1,0}, \pi_{2,0}, \ldots, \pi_{Q,0}) \iff p_{\psi}(L) - L w_0 \min_{\pi_{1,0}, \pi_{2,0}, \ldots, \pi_{Q,0}} \tilde{C}(\pi_{1,0}, \pi_{2,0}, \ldots, \pi_{Q,0}),
\]
therefore the optimal relative quantities \(\pi_{1,0}, \ldots, \pi_{Q,0}\) are obtained from the minimisation of the relative costs \(\tilde{C}(\pi_{1,0}, \pi_{2,0}, \ldots, \pi_{Q,0})\).

\(\tilde{C}(\pi_{1,0}, \pi_{2,0}, \ldots, \pi_{Q,0})\) is a strictly quasi-convex function\(^2\) of the quantities \(\pi_{1,0}, \ldots, \pi_{Q,0}\). Therefore, the first order conditions are sufficient to describe optimality, and they imply that the optimal quantities \(\tilde{\pi}_{p,0}, q=1, \ldots, Q\) must satisfy
\[
\gamma \frac{\phi_q \left(\tilde{\pi}_{q,0}\right)^{\frac{1}{\gamma}}}{1 + \sum_{p=1}^{Q} \phi_p \left(\tilde{\pi}_{p,0}\right)^\gamma} = \frac{\alpha_q \left(\tilde{\pi}_{q,0}\right)^{\frac{1}{\gamma}}}{1 + \sum_{p=1}^{Q} \alpha_p \left(\tilde{\pi}_{p,0}\right)^\gamma}, \quad q=1, \ldots, Q
\]
(18)

In turn, this last expression implies that for any two distinct qualifications \(q\) and \(r\) we must have at the optimum:
\[
\frac{\tilde{\pi}_{q,0}}{\tilde{\pi}_{r,0}}^{\gamma-\rho} = \frac{\alpha_q}{\alpha_r} \iff \tilde{\pi}_{q,0} = \left(\frac{\alpha_q}{\alpha_r} \right)^{\frac{\sigma \tilde{\pi}_{r,0}}{\sigma - \rho}} \phi_{q,0} \phi_{r,0},
\]
and we conclude that \(\tilde{\pi}_{q,0}\) for \(q = 1, \ldots, Q\), must take the form:

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\(^2\) This is a direct consequence of our functional assumptions: \(F(1, \pi_{1,0}, \pi_{2,0}, \ldots, \pi_{Q,0})\) is a strictly concave function, and \(1 + \sum_{q=1}^{Q} \phi_q \left(\pi_{q,0}\right)^\gamma\) is a strictly convex function of the relative proportions. The relative costs function \(C(\pi_{1,0}, \pi_{2,0}, \ldots, \pi_{Q,0})\) is not a convex function in general however (consider the ratio \((1+x^\gamma)/(1+x)^\rho\) with \(\rho < 1 < \gamma, \forall x \geq 0\), for example with \(\rho = 0.5\) and \(\gamma = 1.5\) for \(x \in [0,10]\), see Schaible (1976).
for some positive constant \( \tilde{\Lambda} \) independent of \( q \). We can then show that

**Proposition 1.**

i) The aggregate monopsony employs less of each qualification \( q=1...Q \) than it would under perfect competition on these labour markets, that is \( \tilde{\Lambda} \) is strictly less than 1.

ii) The aggregate monopsony pays relatively less and preserves the competitive wage differentials between qualifications, \( q=1...Q \).

Hence, we can write \( \tilde{\pi}_{q,0} = \tilde{\Lambda}\pi^*_q \). Since \( \tilde{\Lambda} \) is strictly less than 1, we conclude that the monopsonistic production sector sets \( \tilde{\pi}_{q,0} \), \( q=1,...,Q \), such that \( \tilde{\pi}_{q,0} < \pi^*_q \), and therefore \( \tilde{\pi}_0 > \pi^*_0 \). We interpret \( \tilde{\Lambda} \) as a measure of the market power of the monopsony over the labour markets for the Q qualifications. Geometrically the monopsony reduces all relative quantity by a factor \( \tilde{\Lambda} \) along the ray going through the competitive allocation \( \pi^*_q \), \( q=1,...,Q \).

At the optimum the production side pays wages for all qualification which are smaller relative to the wage of the 0th qualification than they were in the competitive setting:

\[
\ln \tilde{w}_{q,0} = \frac{1}{\delta} \ln \tilde{\Lambda} + \frac{1}{\delta} \ln \pi^*_q - \frac{\delta_q}{\delta} = \frac{1}{\delta} \ln \tilde{\Lambda} + \ln \pi^*_q < \ln w^*_q, \quad q=1,...,Q.
\]

i.e. the relative wages are uniformly \( \frac{1}{\delta} \ln \tilde{\Lambda} \) less than the competitive relative wages and we verify that the 0th qualification remains the highest paid qualification, since all the relative wages are less than 1. Hence, the ranking of the monopsony’s relative wages is identical to the ranking of the competitive relative wages, and the monopsony preserves the competitive relative wage differentials. Geometrically the monopsony reduces all relative wages by a factor \( \tilde{\Lambda}^{\frac{1}{\delta}} < 1 \) along the ray going through the competitive relative wages \( w^*_q \), \( q=1,...,Q \).

Given our assumptions, the exercise of market power here does not create any unemployment: the relative quantities are “on the relative supply curves” at the monopsony relative wages. The cost of market power here is in term of misallocation of labour across qualification compared to the competitive distribution. As a result, a larger proportion of the population chooses to obtain the 0th qualification given the wage structure.

Assume that we wish to compare competitive equilibria such that the relative wages and proportions are kept constant in the competitive equilibrium, while changing some of the characteristics of the technology or of the preferences.
That is we imagine changing $\rho$ or $\gamma$ while maintaining $w_{q,0}^*$ and $\pi_{q,0}^*$, $q=1,...,Q$, constant. In such circumstance $A = \sum_{p=1}^{Q} w_{p,0}^* \pi_{p,0}^*$ is kept unchanged by a variation of $\rho$ or $\gamma$. Given this assumption we can conclude:

$$\frac{1}{\lambda} \frac{\partial \hat{\lambda}}{\partial \rho} \bigg|_{\lambda \text{ constant}} = \frac{\hat{\lambda}^{\gamma-\rho} \ln \hat{\lambda}}{1 - \rho \hat{\lambda}^{\gamma-\rho}} \leq 0.$$ 

This quantity is negative for all values of $\rho$. Comparing two labour markets with the same competitive equilibrium distribution of workers across qualifications and identical relative wages, the more substitutable the qualifications (the larger $\rho$) the larger the ability of the monopsony to reduce the relative wages and relative employment. Conversely the smaller $\rho$, i.e. the larger the complementarity between the qualifications, the smaller the monopsony power (the closer $\hat{\lambda}$ is to 1).

A similar calculation this time dealing with $\gamma$ gives:

$$\frac{\gamma}{\hat{\lambda}} \frac{\partial \hat{\lambda}}{\partial \gamma} \bigg|_{\lambda \text{ constant}} = -\frac{\hat{\lambda}^{\gamma-\rho} + A \hat{\lambda}^{\gamma} + \ln \hat{\lambda}}{1 - \rho \hat{\lambda}^{\gamma-\rho}},$$

which we cannot sign in general. Assume that $\gamma \rightarrow 1$ from above, or $\delta \rightarrow +\infty$, in the limit $\hat{\lambda} \rightarrow 1$ and the elasticity is negative for all positive values of $A$. In this case the supply side is very sensitive to changes to the relative wages. Reducing this sensitivity, i.e. reducing $\delta$ which increases $\gamma$, reduces $\hat{\lambda}$ and therefore increases the production sector’s market power.

The production side demands and employs a (strictly) larger proportion of workers with the $0$th qualification when it acts as a monopsony. Furthermore, since the relative wages and the probabilities for each qualification are smaller than the corresponding competitive relative wages and probabilities, the monopsony production sector generates a relative wage distribution for qualifications $q \geq 1$ which is more “compressed” than the one generated in the competitive setting.

Finally we describe the tax policy which corrects for the actions of the “monopsonistic” production sector in our model. We assume that the government can tax/subsidise all relative wages at the margin at a rate $\theta$, $\theta > 0$, and redistribute the costs/subsidies through a lump sum tax. This means that the production sector perceives the relative supply for each qualification to be:

$$w_{q,0}^s = \theta \phi_q \pi_{q,0}^s, \text{ for } q = 1,...,Q,$$

(20)

for some positive value of $\theta$. In effect the minimisation problem with a tax/subsidy $\theta$ has the same general form as the original minimisation problem if we adjust the relative supply functions by a factor $\theta$. The monopsony sets the relative quantities for each qualification such that:
where the quantities with primes denote the optimal quantities set by the monopsony in the presence of the tax/subsidy $\theta$. The last expression arises directly from our earlier analysis by replacing $\phi_q$ with $\theta \phi_q$. The value of the subsidy which leads the monopsony to employ the competitive relative quantities is such that:

$$\tilde{\Lambda}' = \theta^{\gamma-\rho}.$$

Adjusting $A$ to $A' \equiv \theta^{\gamma+\sigma/(1-\sigma)} A$, and applying the argument which proves proposition 1, we find that $\theta = (\gamma + A(\gamma-1))^{-1} < 1$. Hence at the margin the corrective instrument is a subsidy, that is the government pays the worker a share $1-\theta$ of the wage and the monopsony sector pays the other fraction $\theta$ of the wage. This subsidy is financed by a lump sum tax on profits exactly equal to the subsidy value at the competitive level: $(1-\theta)A$. While this solution is simple enough, it is not the one that is used to correct for the effect of market power on the labour market. The next section describes the effect of the minimum wage on all the different labour markets as its relative size increases.

**Monopsony and the Effect of the Minimum Wage**

In this section, we analyse the effect of the minimum wage on the actions of the monopsony. We maintain the assumption that the wage of the highest paid qualification is determined outside of the model all other relative prices being given.

The structure of our model and in particular of the supply curves for each qualification simplifies the analysis substantially. Given the wage for the $0^{th}$ qualification, the introduction of a minimum wage introduces technical difficulties, in particular the profit function is no longer necessarily continuously differentiable: the minimum wage creates a (well known) discontinuity of the marginal cost for each qualification $q=1,\ldots,Q$. In the textbook analysis at the optimum for the monopsony, the introduction of the minimum wage decreases the marginal cost and therefore implies that the employment that maximises profits is larger. In our context, the intuition concerning the marginal cost for each qualification is the same: the introduction of the minimum wage decreases the marginal cost for the newly affected qualification. However it will increase the marginal cost for all other qualifications “already” covered by the minimum wage. Hence the textbook analysis does not extend straightforwardly (even given our functional form assumptions).

We keep the wage of the $0^{th}$ qualification fixed at $w_0$. In our context it is natural to consider $\omega \equiv \frac{w}{w_0}$ the minimum wage relative to $w_0$, for $0 \leq \omega \leq 1$. The relative supplies are increasing in the relevant relative wage, that is for each
qualification the value $\bar{\pi}_{q,0} \equiv (\omega/\phi_q)^{\frac{1}{\phi_q}}$ is such $\bar{\pi}_{q,0} < \pi_{q,0} \Leftrightarrow \bar{w}_{q,0} < \omega$. Finally, note that the relative costs for each qualification

$$\pi_{q,0} w_{q,0} (\pi_{q,0}) = \phi_q \bar{\pi}_{q,0}^{\frac{1}{\phi_q}}, \quad q=1…Q,$$

are strictly increasing and convex functions of the relative supply $\pi_{q,0}$. For a given value of $w_0$ and of the relative minimum wage $0 \leq \omega \leq 1$, the monopsony production sector maximises total profits over $N_0$ and $\pi_{1,0}, \pi_{2,0}, \ldots, \pi_{Q,0}$:

$$p\psi\left(N_0 F\left(1, \pi_{1,0}, \pi_{2,0}, \ldots, \pi_{Q,0}\right) - w_0 N_0 \left(1 + \sum_{q=1}^{Q} \pi_{q,0} \max \left(\omega, \phi_q (\pi_{q,0})^{\frac{1}{\phi_q}}\right)\right)\right). \quad (22)$$

The problem can equivalently be set up as a two stage optimisation problem, whereby the production sector minimises aggregate units costs in terms of relative employment of each qualification and then maximise profit with respect to overall aggregate labour input. Define $L = N_0 F\left(1, \pi_{1,0}, \pi_{2,0}, \ldots, \pi_{Q,0}\right)$ and rewrite the previous expression as

$$p\psi(L) - L w_0 C\left(\omega; \pi_{1,0}, \pi_{2,0}, \ldots, \pi_{Q,0}\right),$$

with

$$C\left(\omega; \pi_{1,0}, \pi_{2,0}, \ldots, \pi_{Q,0}\right) \equiv \frac{1 + \sum_{q=1}^{Q} \pi_{q,0} \max \left(\omega, \phi_q (\pi_{q,0})^{\frac{1}{\phi_q}}\right)}{F\left(1, \pi_{1,0}, \pi_{2,0}, \ldots, \pi_{Q,0}\right)}$$

and we note that

$$\max_{L, \pi_{1,0}, \pi_{2,0}, \ldots, \pi_{Q,0}} p\psi(L) - L w_0 C\left(\omega; \pi_{1,0}, \pi_{2,0}, \ldots, \pi_{Q,0}\right)$$

$$\Leftrightarrow \max_{L} p\psi(L) - L w_0 \cdot \min_{\pi_{1,0}, \pi_{2,0}, \ldots, \pi_{Q,0}} C\left(\omega; \pi_{1,0}, \pi_{2,0}, \ldots, \pi_{Q,0}\right).$$

Hence the effect of the minimum wage on the profit of the monopsony aggregate production sector is determined by the effect of the minimum wage on the minimum of the costs per unit of aggregate labour, $C\left(\omega; \pi_{1,0}, \pi_{2,0}, \ldots, \pi_{Q,0}\right)$. For any given value of $\omega$, $C\left(\omega; \pi_{1,0}, \pi_{2,0}, \ldots, \pi_{Q,0}\right)$ is a strictly quasi-convex function of $\left(\pi_{1,0}, \pi_{2,0}, \ldots, \pi_{Q,0}\right)^3$. Therefore $C\left(\omega; \pi_{1,0}, \pi_{2,0}, \ldots, \pi_{Q,0}\right)$ is minimised at some unique value $\bar{\pi}(\omega) \equiv \left(\bar{\pi}_{1,0}(\omega), \bar{\pi}_{2,0}(\omega), \ldots, \bar{\pi}_{Q,0}(\omega)\right)$. Because of the presence of the minimum wage the first order conditions for the qualification will depend on whether or not a particular group of qualifications is affected by the minimum wage or not. Furthermore, for some qualification the optimal

\[\text{Footnote:} \quad \text{Again, this is a direct consequence of our functional assumptions:} \ F\left(1, \pi_{s,0}, \pi_{2,0}, \ldots, \pi_{Q,0}\right) \text{is a strictly concave function, and} \ 1 + \sum_{q=1}^{Q} \pi_{q,0} \max \left(\omega, \phi_q (\pi_{q,0})^{\frac{1}{\phi_q}}\right) \text{is a convex function of the relative proportions given the relative minimum wage.} \]
quantities are located on a kink of the relative cost function. We can summarise the situation as follows:

i) The optimal solution for qualification \( q \), \( \tilde{\pi}_{q0}(\omega) \), is such that
\[
\tilde{\pi}_{q0}(\omega) < \left( \frac{\omega}{\phi_q} \right) ^{\delta}
\]
and:
\[
\frac{\alpha_q \left( \tilde{\pi}_{q0}(\omega) \right)^{\rho - 1}}{1 + \sum_{k=1}^{Q} \alpha_k \left( \tilde{\pi}_{k0}(\omega) \right)^{\rho}} = \frac{\omega}{1 + \sum_{k=1}^{Q} \tilde{\pi}_{k0}(\omega) \max \left( \omega, \phi_k \left( \tilde{\pi}_{k0}(\omega) \right)^{\frac{1}{\delta}} \right)}.
\]

(23)

ii) The optimal solution for qualification \( q \) is at a kink, \( \tilde{\pi}_{q0}(\omega) = \left( \frac{\omega}{\phi_q} \right) ^{\delta} \) (the relative cost is decreasing to the right of \( \tilde{\pi}_{q0}(\omega) \) and increasing the left of \( \tilde{\pi}_{q0}(\omega) \)) and :
\[
\frac{\omega}{1 + \sum_{k=1}^{Q} \tilde{\pi}_{k0}(\omega) \max \left( \omega, \phi_k \left( \tilde{\pi}_{k0}(\omega) \right)^{\frac{1}{\delta}} \right)} < \gamma \frac{\phi_q \left( \tilde{\pi}_{q0}(\omega) \right)^{\frac{1}{\delta}}}{1 + \sum_{k=1}^{Q} \tilde{\pi}_{k0}(\omega) \max \left( \omega, \phi_k \left( \tilde{\pi}_{k0}(\omega) \right)^{\frac{1}{\delta}} \right)}.
\]

(24)

iii) The optimal solution for qualification \( q \), \( \tilde{\pi}_{q0}(\omega) \), is such that
\[
\tilde{\pi}_{q0}(\omega) > \left( \frac{\omega}{\phi_q} \right) ^{\delta}
\]
(the relative wage on the market for the \( q^{th} \) qualification is larger than the minimum wage) and:
\[
\frac{\alpha_q \left( \tilde{\pi}_{q0}(\omega) \right)^{\rho - 1}}{1 + \sum_{k=1}^{Q} \alpha_k \left( \tilde{\pi}_{k0}(\omega) \right)^{\rho}} = \gamma \frac{\phi_q \left( \tilde{\pi}_{q0}(\omega) \right)^{\frac{1}{\delta}}}{1 + \sum_{k=1}^{Q} \tilde{\pi}_{k0}(\omega) \max \left( \omega, \phi_k \left( \tilde{\pi}_{k0}(\omega) \right)^{\frac{1}{\delta}} \right)}.
\]

(25)

We assume throughout that the quantities \( \alpha_q, \phi_q \) are increasing with \( q = 1, \ldots, Q \), that is the competitive relative wages are increasing with \( q \). We can show the following properties of the solution of this minimisation problem given \( \omega \).

First we show that the minimum wage affects each qualification in turn according to their competitive ranking.
**Proposition 2.**
Assume that the minimum wage $\omega$ does not cover qualification $s$ when the aggregate monopsony maximises profits: then the minimum wage does not bite on all qualifications $r$ with $r \geq s$.

In particular this implies that the smallest value of the relative minimum wage which “bites” (i.e. such that its affects the wage of a single qualification) is equal to the smallest value of the wage paid by the monopsony, $\hat{w}_{10}$.

We turn to the employment effects of the minimum wage, and we show next that when the competitive differentials are large enough, for some values of the minimum wage it is possible that all the qualifications covered by the minimum wage experience unemployment.

**Proposition 3.**
Assume that the minimum wage $\omega$ is such that the first $p$, $p<Q$, qualifications are covered by the minimum wage. A sufficient condition for workers with qualifications $1$ to $p$ to experience unemployment is that the competitive relative wage differentials between the qualifications covered by the minimum wage and the qualifications not covered by the minimum wage are larger than $\frac{1}{\gamma^{1+\sigma}}$.

By implication, proposition 3 suggests that qualifications with small enough competitive relative wage differentials will respond to the minimum wage similarly as far as unemployment is concerned. The bound $\frac{1}{\gamma^{1+\sigma}}$ on the relative wage differentials is decreasing with either parameters $\delta$ or $\sigma$. Consider two independent labour markets with identical competitive wages but different substitution elasticities. The proposition implies that the labour market with the larger substitution elasticity is more likely to exhibit unemployment in the covered qualifications than the other.

Moreover there exists a large enough (relative) minimum wage such that all qualifications are covered and if the relative minimum wage is strictly larger than this value all low paid qualifications will experience unemployment.

**Proposition 4.**
For a large enough relative minimum wage $\omega \geq \bar{\omega}$ all qualifications are covered by the minimum wage. In that case the optimal relative quantities set by the monopsony are such that:

\[
\pi_q(\omega) = \omega^{-\sigma} \alpha_q^\sigma, \quad \text{and} \quad \bar{\omega} = \omega_{q,0}^* = \max_{q=1,...,Q} w_{q,0}^*.
\]

Furthermore if $\omega > \bar{\omega}$, all qualifications with $q>\alpha$ experience unemployment.

Finally we study how the relative proportions for each qualification vary with the minimum wage. Our purpose here is to show what properties on the relative demands that the model structures imposes and to suggest that despite
the strong assumption on the model structure the relative employment responses to an increase in the minimum wage are not “straightforward”: they depend in a complicated way on all parameters in the model. Indeed the optimal relative quantities that the monopsony sector determines for any qualification can be on the demand curve, which will imply the existence of unemployment for that qualification, or on the supply curve.

To proceed, we define the following quantities:

\[
C(\omega) \equiv 1 + \sum_{k=1}^{q} \pi_{k,0}(\omega) \max (\omega, \phi_k (\pi_{k,0}(\omega))^\delta),
\]

\[
\mathcal{F}(\omega) \equiv 1 + \sum_{k=1}^{q} \alpha_{k,0} \pi_{k,0}(\omega),
\]

\[
S_-(\omega) \equiv \sum_{r \in \mathcal{R}} \frac{\omega \pi_{r,0}}{C(\omega)}, \text{ with } S_-(\omega) = 0 \text{ if } \begin{cases} r \in \{1,...,Q\} : \pi_{r,0} < \left(\frac{\omega}{\phi_r}\right)^\delta \end{cases} = \emptyset;
\]

\[
S_\omega(\omega) \equiv \delta \sum_{s \in \mathcal{S}} \left( \frac{\omega \pi_{r,0}}{C(\omega)} - \rho \frac{\alpha_{s,0} \pi_{r,0}^\rho}{\mathcal{F}(\omega)} \right),
\]

with \( S_\omega(\omega) = 0 \) if \( \begin{cases} s \in \{1,...,Q\} : \pi_{r,0} = \left(\frac{\omega}{\phi_s}\right)^\delta \end{cases} = \emptyset \);

\[
S_s(\omega) \equiv \sum_{r \in \mathcal{R}} \phi_r \pi_{r,0}^\gamma \left( \frac{\omega}{\phi_r} \right)^\delta, \text{ with } S_s(\omega) = 0 \text{ if } \begin{cases} t \in \{1,...,Q\} : \pi_{r,0} > \left(\frac{\omega}{\phi_s}\right)^\delta \end{cases} = \emptyset,
\]

and we show:

**Proposition 5.**

Outside of any non-differentiability of the solution with respect to \( \omega \), the elasticity of the relative demands \( \pi_{r,0} \) with respect to \( \omega \) is:

i) for any qualification \( r \) covered by the minimum wage and such that \( \pi_{r,0} < \left(\frac{\omega}{\phi_r}\right)^\delta \) (the monopsony relative quantity for occupation \( r \) is on the “demand curve”):

\[
e_{-\omega} \equiv e_{r,0\omega} = -\sigma \left( \frac{1 - S_-(\omega) + S_\omega(\omega)}{(1 - S_-(\omega)) (\gamma - \rho)} - \frac{\gamma}{\sigma} S_+(\omega) \right), \text{ with } e_{-\omega} \geq -\sigma;
\]

ii) for any qualification \( s \) covered by the minimum wage such that \( \pi_{s,0} = \left(\frac{\omega}{\phi_s}\right)^\delta \) (the monopsony relative quantity for occupation \( s \) is on the “relative supply curve”): \( e_{-\omega} \equiv e_{s,0\omega} = \delta \);
iii) for all qualifications $t$ not covered by the minimum wage, i.e. $\hat{\pi}_{t,0} > \left(\frac{\omega}{\phi_t}\right)^\gamma$ (the monopsony relative quantity for occupation $t$ is on the “relative supply curve”):

$$e_{\omega,t} \equiv e_{t,0t} = \frac{S_\omega(\omega)}{(1-S_\omega(\omega))(\gamma - \rho) - \frac{\gamma}{\sigma} S_\omega(\omega)} , \text{and } e_{t,0t} \geq 0 .$$

Furthermore $e_{\omega,t}$ and $e_{\omega,t}$ are such that:

$$(\gamma - \rho)e_{\omega,t} = 1 + (1 - \rho)e_{\omega,t} .$$

The key finding here is the equality $(\gamma - \rho)e_{\omega,t} = 1 + (1 - \rho)e_{\omega,t}$, which suggests that the elasticities of the relative quantities with respect to the minimum wage across the qualifications that are covered and those that are not, obey a strict relationship. This relationship is clearly the consequence of our functional assumption concerning the qualification choice probabilities and the CES technology.

Finally, these imply that the elasticity of the labour aggregate with respect to the minimum wage is a mixture of the three elasticities $e_{\omega,t}$, $\delta$, $e_{\omega,t}$:

$$e_{\omega} = e_{\omega} \left\{ S_\omega(\omega) - \sum_{k=1}^Q \hat{\pi}_{k,t} \right\} + \delta \left\{ \frac{\sum_{k=1}^Q \pi_{k,t}}{1 + \sum_{k=1}^Q \pi_{k,t}} - \frac{\sum_{k=1}^Q \pi_{x,t}}{1 + \sum_{k=1}^Q \pi_{x,t}} \right\} + e_{\omega} \left\{ S_\omega(\omega) - \sum_{k=1}^Q \hat{\pi}_{k,t} \right\} .$$

In general however we cannot determine the sign of this expression. The proof of the proposition suggests that there exist regimes where $e_{\omega,t}$ is proportional to $e_{\omega,t}$ and $e_{\omega,t}$ can be negative or positive. Hence the effect of the minimum wage on the total employment, as measured by $L = N_\omega F_\omega(\pi_{1,0}, \pi_{2,0}, \ldots, \pi_{Q,0})$, is not necessarily monotonous. Hence, even though our functional form assumptions are restrictive it is interesting to observe that the model is consistent with a variety of qualitative responses to an increase in the relative minimum wage.

Furthermore, Proposition 5 is useful to calculate (numerically) the solution of the optimisation problem for increasing value of the relative minimum wage.

**An Example**

We illustrate the effect of the minimum wage on the economy and in particular on workers’ welfare. To do so we complete the model by specifying $\psi(x)$ as a simple power function:

$$\psi(x) = x^\alpha .$$

In keeping with our assumptions we assume that the market for workers with the $0^{th}$ qualification is competitive, hence that demand for labour is such that the aggregate production sector adjusts the marginal productivity to the marginal costs. Since the relative quantities for all other sectors are obtained
from the minimisation of the relative costs, and given the logit specification for
the supply of workers to each market which depends only on the relative wages,
the supply of workers with \(0^{th}\) qualification is determined from the
minimisation of relative costs. In other words, the aggregate labour input \(\bar{L}\) is
known. The competitive assumption on the market for the \(0^{th}\) qualification
means that the equilibrium wage \(\bar{w}_{0}\) is such that

\[ p\psi'(\bar{L}) = \bar{w}_{0} C(\omega, \pi_{1,0}, \pi_{2,0}, \ldots, \pi_{Q,0}). \]

In all our calculations we assume that \(\alpha = 0.7\) and we normalise \(p\) to 1.

To evaluate the welfare of the population of workers, we use the welfare
measure derived from the logit model evaluated at the relative wages
\(w_{q,0}(\pi_{q,0}(\omega)).\) In the absence of unemployment, our model suggests that the
population’s welfare is (see Anderson, De Palma, & Thisse, 1992)

\[ B(w, b, 0) = E[\max \{I_{0}, I_{1}, \ldots, I_{Q}\}] \]

\[ = \ln \left(\sum_{p=0}^{Q} \exp(\delta_{p} + \delta \ln w_{p})\right) + \gamma \]

\[ = \delta \ln w_{0} + \ln \left(1 + \sum_{p=1}^{Q} \exp(\delta_{p} + \delta \ln w_{p,0})\right) + \gamma, \]

where \(\gamma\) is Euler’s constant\(^4\). The last expression clarifies the role of the relative
wages for qualifications \(q=1,\ldots,Q\) from the role of the wage of the \(0^{th}\)
qualification.

This measure needs to be adapted when unemployment takes place in one or
more markets. We assume that unemployed workers receive a level of benefit \(b,\)
and value unemployment with qualification \(q\) as

\[ I_{q,i}^{h} = \delta_{q} + \delta \ln b + \eta_{q,i}, \]

Note that this implies that \(I_{q,i}^{h} - I_{q,i}^{s} = \delta \ln \frac{w_{i}}{b}\) and this difference does not
depend on the unobservable term \(\eta_{q,i}.\) This is an additional assumption which
we adopt to simplify the welfare calculations. It implies that unemployed
workers who have made a particular qualification choice suffer only from the
opportunity cost of the wage loss relative to the unemployment benefit they
receive.

We consider first the possibility of unemployment in the market for workers
with qualification \(q=1.\) We assume that all workers in that market experience
unemployment with the same probability \(p_{u}.\) The population welfare,
\(B(w, b, p_{u}),\) must be modified to account for the workers who are affected by
the likelihood of unemployment, i.e. if they choose the \(q=1,\) and the others:

\[^4\text{We ignore the effect on welfare of other sources of income.}\]
\[ B(w, b, p_u) = \mathbb{E}\left[ 1_{I' > I_1} I' \right] + (1 - p_u) \mathbb{E}\left[ 1_{I' < I_1} I_1 \right] + p_u \mathbb{E}\left[ 1_{I' < I_1} I_1^b \right] \]
\[ = \mathbb{E}\left[ 1_{I' > I_1} I' \right] + \mathbb{E}\left[ 1_{I' < I_1} I_1 \right] - p_u \pi_1(w)(m_1(w_i) - m_1(b)) \]  \hspace{1cm} \text{(27)}
\[ = B(w, b, 0) - p_u \pi_1(w)(m_1(w_i) - m_1(b)) \]

where \( I' = \max\{I_0, I_2, I_3, \ldots, I_Q\} \), and \( m_q(w_q) \equiv \delta_q + \delta \ln w_q, \ q = 0, 1, \ldots, Q \), and \( m_1(b) \equiv \delta_1 + \delta \ln b \). The first expression distinguishes between individuals who chose \( q=1 \), i.e. such that \( I' < I_1 \), and the others, who are such that \( I' \geq I_1 \).

Among the former group a proportion \( p_u \) is unemployed and the welfare in that case is \( I_1^b \). The second expression uses the property that the difference \( I_1 - I_1^b \) does not depend on the unobservable component \( \eta_{1,i} \) and therefore

\[ p_u \mathbb{E}\left[ 1_{I' < I_1} (I_1 - I_1^b) \right] = \pi_1(w)(m_1(w_i) - m_1(b)). \]

Keeping the wage distribution constant and increasing the unemployment rate for workers with the first qualification the change in indirect utility relative to full employment is:

\[ B(w, b, p_u) - B(w, b, 0) = -p_u \pi_1(w)(m_1(w_i) - m_1(b)), \]  \hspace{1cm} \text{(28)}

which is negative for level of benefits smaller than \( w_1 \).

More generally if the distribution of unemployment across qualifications 1 to \( Q+1 \) is \( p = (p_{11}, p_{22}, \ldots, p_{QQ+1}) \) the population welfare becomes:

\[ B(w, b, p) = B(w, b, 0) - \sum_{q=1}^{Q+1} p_{qq} \pi_q(w)(m_q(w_q) - m_q(b)). \]  \hspace{1cm} \text{(29)}

We set the benefit level to half of the (absolute) minimum wage. A strictly positive value for the benefit allows us to consider the trade-off between a higher minimum wage and a higher level of unemployment. If the benefit level is equal to zero, the costs of unemployment are infinite. This would reduce the usefulness of the minimum wage as a tool against market power. The benefit level is financed with a proportional income tax, \( \tau \), on all income. For a distribution of unemployment in the population the tax \( \tau \) is defined as the ratio of total expenditure on unemployment benefit (the proportion of unemployed workers multiplied by the benefit level) divided by the total income (wage and benefit) in the population. The tax has no consequence on the choices of workers since workers determine the qualification choice on the basis of relative wages only. It has an effect on welfare however, as we can see clearly in equation (26) and (27) where the effect of the proportional income tax on welfare amounts to \( \delta \ln (1-\tau) \). Because the tax is raised on all sources of income it does not have any effect on the welfare costs of unemployment either (given our specification) since they depend on the ratio of the qualification wage to the benefit level. This ratio does not vary with a proportional tax on all income.

We consider \( Q=4 \) low paid qualifications, and we set the model parameters as follows:
\((\alpha_1, \alpha_2, \ldots, \alpha_Q) = (0.001, 0.05, 0.15, 0.20)\),
\((\delta_1, \delta_2, \ldots, \delta_Q) = (0.01, 0.1, 0.5, 0.65)\),
\(\delta = 1.75\) and \(\sigma = 1.75\). Hence the supply side is responsive to relative wage changes, while the technology exhibits substantial complementarities between labour inputs. The competitive wage differentials with the competitive qualification are substantial: \(w'_0 = (0.032, 0.22, 0.34, 0.37)\). These differentials are large enough for unemployment to arise for some qualifications for some values of the relative minimum wage. The value of the competitive wage for the 0th qualification is \(\bar{w}_0 = 0.927\), and the equilibrium distribution across qualification gives a large weight to the competitive sector \(\pi^* = (0.60, 0.0014, 0.046, 0.15, 0.20)\). As we expect the relatively large substitution elasticity gives substantial market power to the production sector, i.e. \(\Lambda = 0.636\) and the relative wages for the low paid qualifications are reduced to 77.2% of their competitive values.

These particular values allow us to illustrate the (possibly) surprising non monotonic effect of the minimum wage on welfare. The top right hand pane in
Figure 1 shows the proportion demanded and supplied with each qualification. Each qualification is represented by a single curve for small values of the relative minimum wage, in this case the demand matches the supply. As the size of the relative minimum wage increases, the demand for the qualification is less than the supply, i.e. there is unemployment in the market for that qualification. The curve for the qualification splits into two curves, the top one represents the proportion supplied while the one below represents the proportion demanded of each qualification. When the supply is larger than the demand the minimum wage bites, whereas when the supply and the demand match the relative wages are determined by the relative supply curves.

The top right hand pane describes the wage of the $$0^{th}$$ qualification while middle left hand pane shows the relative wages as a function of the relative minimum wage. Since the first qualification contributes little to the production a larger minimum wage does affect the demand for the other qualification much. As the minimum wage increases, the wage of the $$0^{th}$$ qualification increases steadily. Intuitively this maintains the size of the wage differentials and ensures that the supply of workers with the $$0^{th}$$ qualification is large enough. As we noticed earlier, the increase of $$w_0$$ contributes positively to the population overall welfare. The variance between the relative wages of the low paid qualification shrinks as the minimum wage increases, and for relatively large minimum wages all relative wages are equal. This has a further positive effect on welfare as shown in the middle right hand pane.

The minimum wage creates unemployment, and unemployment has an increasing welfare cost however, this is illustrated for each qualification in the bottom left hand pane. For each qualification we draw the proportions supplied and demanded. As the minimum wage increases the demand eventually falls below the supply and at that point the schedule for each qualification splits into two curves. The lowest right hand pane shows how welfare varies overall. The total welfare schedule is multimodal. Consider the first local maximum. This arises because the workers with the $$2^{nd}$$ qualification first mostly gain from the increase of the minimum wage and beyond a certain level as unemployment increases in this market the welfare gains are overtaken by the welfare losses that unemployed workers face. The second more pronounced maximum arises for similar reasons when the market for the $$4^{th}$$ qualification is affected by the minimum wage. Total welfare is maximised when the relative minimum wage reaches 33.24% of the $$0^{th}$$ qualification wage.

The aggregate profits (of the aggregate monopsony) on the other hand fall monotonously with the minimum wage (not shown) while the required tax rate increases to capture more than 65% of all income when the minimum wage is equal to the competitive wage for the $$0^{th}$$ qualification. This pattern is robust to changing the substitution elasticity to $$\sigma = 0.25$$.

The competitive relative wages are then $$w^*_0 = (0.42, 0.59, 0.61, 0.65)$$, and the competitive wage for the $$0^{th}$$ qualification is $$w^*_0 = 0.35$$. The workers are distributed among the qualifications such that...
$\pi^* = (0.31, 0.069, 0.24, 0.22, 0.16)$. When the production sector behaves as a monopsony on the low paid qualifications, it sets $\tilde{\lambda}$ to $\tilde{\lambda} = 0.776$ and reduces the relative share of qualifications 1 to 4 to 77.6% of their competitive values and employs 36.5% of the workers in the market for the $0^{th}$ qualification. Furthermore all relative wages are about 86.5% of the competitive relative wages and $0^{th}$ qualification workers are paid $\tilde{w}_0 = 0.366$. The relative minimum wage that maximises overall welfare is 38.69% of the $0^{th}$ qualification wage. The second (local) maxima arises when the relative minimum wage is 56.8% of the $0^{th}$ qualification wage but yields a lower overall welfare value.
Figure 1 The effect of the minimum wage on welfare and costs
Exemptions from the Minimum Wage

We complete our analysis by commenting on the effect of an exemption from the minimum wage for the relative demand of a particular qualification. Exemptions of this kind exist, for example in the UK apprentices were exempt of a minimum wage until 2012 and thereafter the rate that applies to them is about 50% of the regular rate. Clearly, our model does not capture all the relevant considerations that were part of the argument for exemption and against the full rate. The UK rate was set at a distinct level so as to compensate firms (usually small firms) for the costs of training their apprentices. Nor does the model account for the long run wage gains that qualified apprentices will capture.

Assume qualification 1 only is exempt from the coverage of the minimum wage. The aggregate production sector maximises its profits:

$$p\psi\left(N_0F\left(1,\pi_{1,0},\pi_{2,0},...,\pi_{Q,0}\right)\right)-w_0N_0\left(1+\phi_1\pi_{1,0}+\sum_{q=2}^{Q}\pi_{q,0}\max\left(\omega,\phi_q\left(\pi_{q,0}\right)^\frac{1}{\gamma}\right)\right),$$

with respect to $N_0$ and $\pi_{1,0},\pi_{2,0},...,\pi_{Q,0}$, given value of $w_0$ and of the relative minimum wage $0 \leq \omega \leq 1$. Following the analysis described in the previous section, in order to determine the optimal distribution of employment among qualifications the production sector requires to minimise the new relative costs function $C_{-1}\left(\omega;\pi_{1,0},\pi_{2,0},...,\pi_{Q,0}\right)$, with respect to $\pi_{1,0},\pi_{2,0},...,\pi_{Q,0}$, where

$$C_{-1}\left(\omega;\pi_{1,0},\pi_{2,0},...,\pi_{Q,0}\right) \equiv \frac{1+\phi_1\pi_{1,0}+\sum_{q=2}^{Q}\pi_{q,0}\max\left(\omega,\phi_q\left(\pi_{q,0}\right)^\frac{1}{\gamma}\right)}{F\left(1,\pi_{1,0},\pi_{2,0},...,\pi_{Q,0}\right)}.$$

Denote $\bar{\pi}(\omega)$ the vector of relative quantities which minimises $C_{-1}(\omega, \pi)$.

The first order conditions for the optimal quantities for qualifications $q=2,...,Q$, are almost identical to the one we describe in equations (23), (24) and (25), the expressions involving the costs must recognize the exemption from the minimum wage for the 1st qualification. Because of the exemption the optimal quantity for the first qualification, $\bar{\pi}_{1,0}(\omega)$, satisfies:

$$\frac{\alpha_1\left(\bar{\pi}_{1,0}(\omega)\right)^{\rho-1}}{1+\sum_{k=1}^{Q}\alpha_k\left(\bar{\pi}_{k,0}(\omega)\right)^{\rho}} = \frac{\gamma\phi_1\left(\bar{\pi}_{1,0}(\omega)\right)^{\frac{1}{\gamma}}}{1+\phi_1\pi_{1,0}+\sum_{q=2}^{Q}\pi_{q,0}(\omega)\max\left(\omega,\phi_q\left(\pi_{q,0}(\omega)\right)^{\frac{1}{\gamma}}\right)},$$

(30)

where $\bar{\pi}_{q,0}(\omega)$, $q=1..Q$, denotes the optimal quantities for a given relative minimum wage $\omega$.

The exemption of one qualification from the minimum is not a marginal change to the optimisation problem we consider earlier on. It is clear that the relative costs to the monopsony production sector are smaller with the exemption than without, however it is unclear what the effect of any exemption on the allocation across qualifications should be in general. Our approach here is to show the effect of a change of the relative minimum wage when one single qualification is exempted.
We focus on the situation with \( Q = 3 \). This is the first case where we can observe simultaneously a qualification exempt from the minimum wage (\( q = 1 \)), one qualification (\( q = 2 \)) where the minimum wage bites, and lastly one qualification (\( q = 3 \)) not covered by the exemption where the minimum wage does not bite.

We assume that \( \omega \) is such that 
\[
\left( \begin{array}{c}
\omega \\
\phi_2 \\
\phi_3
\end{array} \right) \leq \left( \begin{array}{c}
0 \\
0 \\
0
\end{array} \right)
\]
and
\[
\left( \begin{array}{c}
\omega \\
\phi_2 \\
\phi_3
\end{array} \right) \geq \left( \begin{array}{c}
0 \\
0 \\
0
\end{array} \right)
\]

The first order condition which determines \( \bar{\pi}_{1,0}(\omega) \) and \( \bar{\pi}_{3,0}(\omega) \) demands that
\[
\bar{\pi}_{1,0}(\omega) = \frac{\alpha_1 \phi_3}{\phi_1 \alpha_3} \gamma \gamma - \rho \bar{\pi}_{3,0}(\omega),
\]
since we demand \( \bar{\pi}_{3,0}(\omega) > \left( \frac{\omega}{\phi_3} \right)^{\delta} \) this implies
\[
\bar{\pi}_{1,0}(\omega) > \left( \frac{\alpha_1 \phi_3}{\phi_1 \alpha_3} \gamma \gamma - \rho \right) \left( \frac{\omega}{\phi_3} \right)^{\delta} = \left( \frac{w_{1,0}^*}{w_{3,0}^*} \right)^{\delta} \left( \frac{\omega}{\phi_3} \right)^{\delta}.
\]

Our earlier assumptions imply that the relative wages are ordered such that \( w_{1,0}^* < w_{3,0}^* \). Hence the first order conditions in the optimisation with the exemption do not require for the relative demand for workers with the first qualification to be larger than what would be supplied if the minimum wage applied to the first qualification.

The following proposition describes how the relative minimum wage affects the demand for the first and third qualification assuming that the change is such that the 2\(^{nd}\) qualification remains on its supply curve.

**Proposition 6.**
Under the previous assumptions, the relative labour demand on the markets for the 1\(^{st}\) and 3\(^{rd}\) qualifications respond identically to changes of the relative minimum wage and their demand elasticity w.r.t. \( \omega \) is always positive and is equal to
\[
e_{1,0,\omega} = e_{3,0,\omega} = \frac{\delta}{\phi_2 \bar{\pi}_{2,0} - \rho \alpha_2 \bar{\pi}_{2,0} \bar{\Lambda}^{\gamma - \rho}} \left( \gamma - \rho \right) \left( 1 + \alpha_2 \bar{\pi}_{2,0} \right) \bar{\Lambda}^{\gamma - \rho} + (\gamma - 1) A_{-1} \Lambda^{\gamma} < \delta,
\]
where \( \bar{\Lambda} \) solves:
\[
\gamma \left( 1 + \alpha_2 \bar{\pi}_{2,0} \right) \bar{\Lambda}^{\gamma - \rho} + (\gamma - 1) A_{-1} \bar{\Lambda}^{\gamma} = 1 + \phi_2 \bar{\pi}_{2,0}.
\]
Furthermore we have \( \bar{\pi}_{1,0}(\omega) \leq \left( \frac{\omega}{\phi_1} \right)^{\delta} \) and the first qualification is paid less than the relative minimum wage.

Hence as long as we remain on the supply curve for the 2\(^{nd}\) (covered) occupation an increase of the minimum wage increases the demand for the other low paid occupations. In response to the exemption the monopsony production sector responds to the effect on supply of setting a relative wage less
than the relative minimum wage: it depresses the relative supply of workers with the first qualification compared to the supply of the other qualifications. As the relative minimum wage increases, and provided that the qualitative features of the solution do not change, the monopsony production sector increases its demand for all qualifications \( q=1\ldots 3 \), and this increase does not create any unemployment. Note however that while the elasticity of the relative supply for the 2\(^{nd}\) qualification w.r.t. the relative minimum wage is equal to \( \delta \), the elasticity of the demand for the other two qualifications is less than \( \delta \).
Conclusion

The conclusions of the analysis of a partial equilibrium labour market model with a monopsony are not robust to its extension to a model where workers are able to choose between several labour markets. For example, our analysis shows that the presence of market power need not generate unemployment, although it will generate welfare losses relative to the competitive outcome. The welfare cost of market power arises here because workers are not allocated to the qualifications which would have been preferred in the competitive world. Hence the market power we consider here creates a misallocation cost.

Our model ignores the interactions between market segmentation on the one hand and other characteristics of the labour market on the other hand, like search and matching (see Flinn, 2010) or on-the-job training (as analysed for example in Acemoglu & Pischke, 2003), and modifying the model to account for such features these will no doubt modify our results.

In our particular framework, the introduction of a minimum wage is not the best corrective policy, we show in particular that a mixture of a wage subsidy and lump sum taxation of the production sector will provide a “simple” remedy to the existence of market power. This was an expected result given the recent literature on the role of the minimum wage in an optimal taxation framework (see Cahuc & Laroque, 2014 and Lee & Saez, 2012).

The interest in our analysis can be found in the analysis of the effect of the minimum wage on the wage and employment distribution relative to the (better paid) competitive sector. We show that a large enough minimum wage creates unemployment for all qualifications but the qualification that is traded competitively. For moderate values of the minimum wage the possibility of unemployment exists for the qualifications which are covered by the minimum wage. This depends on the particular parameter describing the supply response to a change in the wage. We expect unemployment among the qualifications covered by the minimum wage if the wage differentials between the competitive wage of the covered qualification and the first qualification not covered are large enough. The necessary size for this quantity depends simply in our model on the parameter of the wage in the supply functions. The larger the sensitivity of the supply for qualification to the wage, the smaller the competitive wage differential required to lead to unemployment when a moderate minimum wage is introduced.

The effect of the minimum wage on aggregate employment is ambiguous in general, although we can exactly describe its elasticity with respect to the minimum wage in each regime. Precise results require further assumptions about the nature of the technology and the tastes. While this is a difficulty for the theorist, for the empiricist is may be an advantage: the theoretical model does not impose a constraining structure a priori.

Despite its apparent simplicity, the theoretical analysis indicates that any empirical study of the effect of the minimum wage on the supply of qualification requires a model of the proportions employed given the qualifications which allows for a variable and heterogeneous response to the
minimum wage. Our analysis shows that the effect of the minimum wage depends on the number of sectors affected, on their particular productivities, as well as on the substitution elasticity and on the supply side characteristics.

We show furthermore that in our model the population welfare is not necessarily a unimodal function of the minimum wage. This is a conclusion of some importance for policy: the optimal minimum wage may be one that creates substantial unemployment.

Finally we describe the effect of an exemption from the minimum wage for lowest paid qualification. As could be expected the exempted segment will behave like the segments that are not covered by the minimum wage, however the effect of an increase of the minimum wage is not limited to the sectors covered by it: the exempted segment will experience a change in the demand for work as the minimum wage increases. Hence the presence of an exemption does insulate the demand for labour in that sector from the effect of the minimum wage in general.
References
Appendix: Proofs for the propositions in the text.

Proof of Proposition 1
To determine the value of \( \Lambda \), we multiply the first order conditions, for each \( q=1,...,Q \), by \( \bar{\pi}_{q,0} \) and sum over \( q \), this provide an additional restriction on the relative proportions such that:

\[
\gamma \frac{\sum_{p=1}^{Q} \phi_p \left( \bar{\pi}_{p,0} \right) \gamma}{1 + \sum_{p=1}^{Q} \phi_p \left( \bar{\pi}_{p,0} \right) \gamma} = \frac{\sum_{p=1}^{Q} \alpha_p \left( \bar{\pi}_{p,0} \right) \rho}{1 + \sum_{p=1}^{Q} \alpha_p \left( \bar{\pi}_{p,0} \right) \rho}.
\]

Furthermore we observe that

\[
\sum_{p=1}^{Q} \alpha_p \left( \bar{\pi}_{p,0} \right) \rho = \Lambda^\rho A,
\]

\[
\sum_{p=1}^{Q} \phi_p \left( \bar{\pi}_{p,0} \right) \gamma = \Lambda^\gamma A,
\]

where \( A \equiv \sum_{p=1}^{Q} \left( \frac{1}{\phi_p} \right)^{\delta(\sigma-1)} \left( \alpha_p \right)^{\frac{\sigma(\sigma+1)}{\delta+\sigma}} \left( \bar{\pi}_{p,0} \right)^{\frac{\sigma(\delta+1)}{\delta+\sigma}} = \sum_{p=1}^{Q} w^*_p \bar{\pi}^*_p \), the competitive costs of employing all qualifications \( q=1...Q \) relative to the wage bill costs of the 0th qualification.

Together with our first expression these observations imply that \( \tilde{\Lambda} \) satisfies:

\[
\gamma \tilde{\Lambda}^{\gamma-\rho} + (\gamma - 1) \tilde{\Lambda}^{\gamma} A - 1 = 0
\]

(31)

As a function of its argument, say \( \lambda \), the left hand side of the previous expression is increasing with \( \lambda \) (for any value of the parameters such that \( 1 + \frac{1}{\delta} > \rho = 1 - \frac{1}{\sigma} \)) and takes the value -1 when \( \lambda = 0 \) and is equal to \( (\gamma - 1)(A + 1) \) for \( \lambda = 1 \). Hence the root \( \tilde{\Lambda} \) is less than 1. We verify further that \( \tilde{\Lambda} \) must be less than \( \left( \frac{1}{\gamma} \right)^{\gamma-\rho} < 1 \).

Moreover, we can write \( \bar{\pi}_{q,0} = \tilde{\Lambda} \bar{\pi}^*_q \). Since \( \tilde{\Lambda} \) is strictly less than 1, we conclude that the monopsonistic production sector sets \( \bar{\pi}_{q,0}, \ q=1,...,Q \), such that \( \bar{\pi}_{q,0} < \bar{\pi}^*_q \), and therefore \( \bar{\pi}_0 > \bar{\pi}^*_0 \). We interpret \( \tilde{\Lambda} \) as a measure of the market power of the monopsony over the labour markets for the Q qualifications. Geometrically the monopsony reduces all relative quantity by a factor \( \tilde{\Lambda} \) along the ray going through the competitive allocation \( \bar{\pi}^*_q, \ q=1,...,Q \).
At the optimum the production side pays wages for all qualification which are smaller relative to the wage of the \(o\)th qualification and demands fewer of each qualification relative to the \(o\)th qualification since

\[
\ln w_{q,0} = \frac{1}{\delta} \ln \bar{\Lambda} + \frac{1}{\delta} \ln \pi^*_q - \frac{\delta_q}{\delta} = \frac{1}{\delta} \ln \bar{\Lambda} + \ln w^*_q < \ln w^*_q \leq 0
\]

i.e. the relative wages are uniformly \(\frac{1}{\delta} \ln \bar{\Lambda}\) less than the competitive relative wages and we verify that the \(o\)th qualification remains the highest paid qualification, since all the relative wages are less than 1. Hence, the ranking of the monopsony’s relative wages is identical to the ranking of the competitive relative wages, and the monopsony preserves the competitive wage differentials. Geometrically the monopsony reduces all relative wage by a factor \(\frac{1}{\bar{\Lambda}^\frac{1}{\delta}} < 1\) along the ray going through the competitive relative wages \(w^*_q\), \(q=1,\ldots,Q\).

**Proof of Proposition 2**

Given the relative minimum wage \(\omega\), consider any two qualifications \(r\) and \(s\) which satisfy first order conditions of the minimisation of the relative costs and the optimal quantities \(\pi_{r,0}(\omega)\) and \(\pi_{s,0}(\omega)\) are such that \(\pi_{s,0}(\omega) \geq \left(\frac{\omega}{\phi_s}\right)^{\frac{1}{\gamma}}\) and \(\pi_{r,0}(\omega) \geq \left(\frac{\omega}{\phi_r}\right)^{\frac{1}{\gamma}}\). This implies that

\[
\frac{\alpha_s}{\alpha_r} \left(\frac{\pi_{s,0}(\omega)}{\pi_{r,0}(\omega)}\right)^{\frac{1}{\gamma}} = \left(\frac{\alpha_s}{\phi_s} \left(\frac{\pi_{s,0}(\omega)}{\phi_r} \left(\frac{\pi_{s,0}(\omega)}{\phi_s} \right)^{\frac{1}{\gamma}}\right)^{\frac{1}{\gamma}}\right) \leftrightarrow \pi_{r,0}(\omega) = \pi_{s,0}(\omega) \left(\frac{\alpha_r}{\phi_r} \left(\frac{\pi_{s,0}(\omega)}{\phi_s} \right)^{\frac{1}{\gamma}}\right)^{\frac{1}{\gamma}},
\]

Therefore we have

\[
\pi_{r,0}(\omega) = \pi_{s,0}(\omega) \left(\frac{\alpha_r}{\phi_r} \left(\frac{\pi_{s,0}(\omega)}{\phi_s} \right)^{\frac{1}{\gamma}}\right)^{\frac{1}{\gamma}} \geq \left(\frac{\omega}{\phi_r}\right)^{\frac{1}{\gamma}} \quad \text{and} \quad \pi_{s,0}(\omega) \geq \left(\frac{\omega}{\phi_s}\right)^{\frac{1}{\gamma}}
\]

\[
\left(\frac{\omega}{\phi_s}\right)^{\frac{1}{\gamma}} \left(\frac{\alpha_r}{\pi_{s,0}(\omega)} \left(\frac{\pi_{s,0}(\omega)}{\phi_s} \right)^{\frac{1}{\gamma}}\right)^{\frac{1}{\gamma}} \geq \left(\frac{\omega}{\phi_r}\right)^{\frac{1}{\gamma}} \quad \text{and} \quad \pi_{s,0}(\omega) \geq \left(\frac{\omega}{\phi_s}\right)^{\frac{1}{\gamma}}
\]

\[
\leftrightarrow \alpha_r^\gamma \phi^\gamma_r \geq \alpha_s^\gamma \phi^\gamma_s \quad \text{and} \quad \pi_{s,0}(\omega) \geq \left(\frac{\omega}{\phi_s}\right)^{\frac{1}{\gamma}},
\]

i.e if qualification \(s\) demands more than the quantity supplied given the relative minimum wage and therefore pays more than the relative minimum wage then all qualifications above the \(s\)th qualification (in the sense \(r \geq s\)) will be paid more than the relative minimum wage.

**Proof of Proposition 3**

For some relative minimum wage \(\omega\) the statement suggests that first order conditions for optimality imply:
\[
\frac{\alpha_q \left( \bar{\pi}_{r,0} (\omega) \right)^\rho}{1 + \sum_{k=1}^{q} \alpha_k \left( \bar{\pi}_{k,0} (\omega) \right)^\rho} = \frac{\omega \bar{\pi}_{q,0} (\omega)}{1 + \sum_{k=1}^{q} \bar{\pi}_{k,0} (\omega) \max \left( \omega, \phi_k \left( \bar{\pi}_{k,0} (\omega) \right)^\frac{1}{\gamma} \right)}
\]

for all \( q \leq p \)

and

\[
\frac{\alpha_q \left( \bar{\pi}_{u,0} (\omega) \right)^\rho}{1 + \sum_{k=1}^{q} \alpha_k \left( \bar{\pi}_{k,0} (\omega) \right)^\rho} = \frac{\gamma}{1 + \sum_{k=1}^{q} \bar{\pi}_{k,0} (\omega) \max \left( \omega, \phi_k \left( \bar{\pi}_{k,0} (\omega) \right)^\frac{1}{\gamma} \right)}
\]

for all \( q > p \).

These imply that for any qualification \( r \) in the first group and any qualification \( u \) in the second group, \( r \leq p + 1 \leq u \), we have:

\[
\bar{\pi}_{r,0} (\omega) < \left( \frac{\omega}{\phi_r} \right)^\delta \quad \text{and} \quad \bar{\pi}_{u,0} (\omega) > \left( \frac{\omega}{\phi_u} \right)^\delta
\]

\[
\frac{\alpha_r}{\alpha_u} \left( \bar{\pi}_{r,0} (\omega) \right)^\rho = \frac{\omega \left( \bar{\pi}_{r,0} (\omega) \right)^1}{\gamma \bar{\pi}_{r,0} (\omega)^\gamma}
\]

for \( r \leq p < p + 1 \leq u \)

which is equivalent to:

\[
\bar{\pi}_{r,0} (\omega) < \left( \frac{\omega}{\phi_r} \right)^\delta \quad \text{and} \quad \bar{\pi}_{u,0} (\omega) > \left( \frac{\omega}{\phi_u} \right)^\delta
\]

\[
\bar{\pi}_{u,0} (\omega) = \left( \frac{\omega \alpha_u}{\gamma \alpha_r \phi_u} \right)^\frac{1}{1-\rho} \left( \bar{\pi}_{r,0} (\omega) \right)^\frac{\gamma-1}{\alpha_r (\omega)} \geq \left( \frac{\omega}{\phi_u} \right)^\delta
\]

for \( r \leq p < u \)

\[
\Leftrightarrow \alpha_u \phi_u \left( \bar{\pi}_{r,0} (\omega) \right)^{\frac{\delta}{\gamma-1}} > \gamma \alpha_r (\omega)^{\delta (\gamma-1)}
\]

\[
\Leftrightarrow \alpha_u \phi_u \left( \bar{\pi}_{r,0} (\omega) \right)^{\frac{\delta}{1-\rho}} > \gamma \alpha_r (\omega)^{\delta}
\]

The LHS of the last inequality is increasing with \( \bar{\pi}_{r,0} (\omega) \), hence for the property to hold for some \( \omega \) it is sufficient that the inequality holds for \( \left( \frac{\omega}{\phi_r} \right)^\delta \) the strict upper bound for \( \bar{\pi}_{r,0} (\omega) \) in this regime (if it is not satisfied the property cannot hold for any value of \( \omega \)). In that case we have:

\[
\alpha_u \phi_u \left( \frac{\omega}{\phi_r} \right)^{\delta} > \gamma \alpha_r (\omega)^{\delta} \Leftrightarrow \alpha_u \phi_u ^\delta > \gamma \alpha_r ^\delta \phi_r ^\delta \quad \text{for} \quad r \leq p < u.
\]

**Proof of Proposition 4**

If the minimum wage bites for all qualifications, then the first order conditions (23) imply that for any two qualifications \( r \) and \( s \) we have, \( r \) and \( s \geq 1 \):

\[
\frac{\bar{\pi}_{r,0}}{\bar{\pi}_{s,0}} \left( \frac{\alpha_r}{\alpha_s} \right)^\rho
\]

From which we deduce that

\[-33-\]
\[ \bar{\pi}_{r,0} = \lambda \alpha_r^\sigma, \] for some constant \( \lambda \).

The first order conditions (23) again imply that \( \lambda \) must satisfy the equation:

\[ \frac{\lambda^\rho B}{1 + \lambda^\rho B} = \frac{\omega \lambda B}{1 + \omega \lambda B} \Rightarrow \lambda = \omega^{-\sigma}, \]

with \( B \equiv \sum_{q=1}^{Q} \alpha_q^\sigma \). Hence the relative shares are

\[ \bar{\pi}_{q,0}(\omega) = \omega^{-\sigma} \alpha_q^\sigma \text{ for } \omega \geq \omega_r, \text{ q=1,...,Q}. \]

We can verify that for all qualifications to be covered by the relative minimum wage we must have

\[ \bar{\pi}_{q,0}(\omega) = \omega^{-\sigma} \alpha_q^\sigma \leq \left( \frac{\omega}{\phi_q} \right)^\delta \text{ for any } \omega \geq \omega_r \text{, for any q=1,...,Q}. \]

Hence

\[ \omega \geq \alpha_q^{\sigma+\delta} \phi_q^{\delta+\sigma}, \text{ for any } q = 1,...,Q. \]

The RHS of this last inequality is the competitive wage for each qualification, and the minimum wage covers all qualifications if the relative minimum wage is larger than the relative wage for the best paid qualification, i.e. \( w_{Q,0}^* \).

If the quantities \( \alpha_q^{\sigma} \phi_q^{\delta}, q=1,...,Q \) are all distinct then some qualifications will experience unemployment. Indeed for two qualifications \( r \) and \( s \) to be on their relative supply curve means at the monopsony optimum means:

\[ \bar{\pi}_{r,0}(\omega) = \left( \frac{\omega}{\phi_r} \right)^\delta \text{ and } \bar{\pi}_{s,0}(\omega) = \left( \frac{\omega}{\phi_s} \right)^\delta \]

and this implies that \( \alpha_r^{\sigma} \phi_r^{\delta} = \alpha_s^{\sigma} \phi_s^{\delta} \), which we ruled out, hence either

\[ \bar{\pi}_{r,0}(\omega) \left( \frac{\omega}{\phi_r} \right)^\delta \text{ or } \bar{\pi}_{s,0}(\omega) \left( \frac{\omega}{\phi_s} \right)^\delta \text{ or both.} \]

Observe further that \( \bar{\pi}_{q,0}(\omega) \) is decreasing with \( \omega \), while the relative supply at the minimum wage is increasing with \( \omega \) for \( \omega \geq w_{Q,0}^* \). Since by assumption

\[ w_{Q,0}^* \equiv \alpha_q^{\sigma+\delta} \phi_q^{\sigma+\delta} < 1, \text{ for any relative minimum wage strictly larger than } w_{Q,0}^*, \text{ the competitive wage of the Q\textsuperscript{th} qualification, all qualifications experience some unemployment (i.e. such that the supply is larger than the demand for labour for each qualification).} \]

**Proof of Proposition 5**

We assume here that the change in the relative minimum does not change the qualitative feature of the solution (the qualification covered and covered by the minimum wage are not changed by an infinitesimal change of the minimum wage).
wage). Hence, the first order conditions determine the effect of the relative minimum wage. In the first case, i), we have:

\[ \frac{\alpha_{r,0}\tilde{\pi}_{r,0}^\rho}{\mathcal{F}(\omega)} = \frac{\omega\tilde{\pi}_{r,0}}{\mathcal{C}(\omega)} \iff \alpha_{r,0}\tilde{\pi}_{r,0}^\rho \mathcal{C}(\omega) = \omega\tilde{\pi}_{r,0}\mathcal{F}(\omega), \]

and we note that the quantities \( \mathcal{C}(\omega) \) and \( \mathcal{F}(\omega) \) are independent of \( r \).

Direct calculations give:

\[ \frac{d\tilde{\pi}_{r,0}}{d\omega} \left\{ \rho\alpha_{r,0}\tilde{\pi}_{r,0}^{\rho-1}\mathcal{C}(\omega) - \omega\mathcal{F}(\omega) \right\} = \tilde{\pi}_{r,0}\mathcal{F}(\omega) + \omega\tilde{\pi}_{r,0}\frac{d\mathcal{F}(\omega)}{d\omega} - \alpha_{r,0}\tilde{\pi}_{r,0}^\rho \frac{d\mathcal{C}(\omega)}{d\omega} \]

\[ \iff \omega \frac{d\tilde{\pi}_{r,0}}{\tilde{\pi}_{r,0}} \frac{d\tilde{\pi}_{r,0}}{d\omega} (\rho - 1)\tilde{\pi}_{r,0}\mathcal{F}(\omega) = \tilde{\pi}_{r,0}\mathcal{F}(\omega) \left\{ 1 + e_{t,\omega} - e_{c,\omega} \right\}, \]

where the second expression accounts for the initial first order conditions in this regime. For all qualifications covered by the minimum wage the elasticities of the relative quantities are clearly equal: \( e_{r,\omega} = e_{c,\omega} \) for all \( r \). Finally the expression simplifies to:

\[ (1 - \rho)e_{c,\omega} = (e_{c,\omega} - e_{g,\omega}) - 1. \quad \text{(32)} \]

Similar calculations can be carried out for case ii) such that \( \tilde{\pi}_{r,0} > \left( \frac{\omega}{\phi_1} \right)^\delta \)

starting from:

\[ \alpha_{r,0}\tilde{\pi}_{r,0}^\rho \mathcal{C}(\omega) = \gamma\phi_1\tilde{\pi}_{r,0}\mathcal{F}(\omega), \]

and in this case the elasticities of the relative quantities not covered by the minimum wage are all equal: \( e_{t,\omega} = e_{c,\omega} \) for all \( t \) which leads to:

\[ (\gamma - \rho)e_{c,\omega} = e_{c,\omega} - e_{g,\omega}. \quad \text{(33)} \]

ii) is immediate, as long as the change in \( \omega \) does not cause a change in the qualitative structure of the optimal solution for the monopsony, the variation in employment is along the relative supply curve which has an elasticity w.r.t \( \omega \) of \( \delta \).

To derive the exact expressions in the proposition we observe that:

\[ e_{c,\omega} = (1 + e_{c,\omega})S_-(\omega) + \delta \sum_{s \neq t} \gamma \frac{\omega\tilde{\pi}_{s,0}}{\mathcal{C}(\omega)} e_{c,\omega}S_+(\omega), \]

\[ e_{g,\omega} = \rho \sum_{p=1}^{g+1} e_{c,\omega} e_{c,\omega} \]

\[ = \rho e_{c,\omega}S_-(\omega) + \rho \delta \sum_{s \neq t} \frac{\alpha_{s,0}\tilde{\pi}_{s,0}^\rho}{\mathcal{F}(\omega)} + \rho e_{c,\omega}S_+(\omega). \]
Where the terms $S_-$ and $S_+$ appear in the second expression because of the first order conditions for the minimisation of the relative costs. Hence, their difference becomes:

$$e_{\omega} - e_{\eta, \omega} = \left(1 + \frac{1}{\sigma} e_{\omega}\right) S_-(\omega) + S_+(\omega) + \frac{\gamma}{\sigma} e_{\eta, \omega} S_+(\omega) \geq 0.$$  

The expression in the proposition are then obtained by solving equations (32) and (33) for $e_{\omega}$ and $e_{\eta, \omega}$. Hence the elasticities of the relative quantities $\pi_{p,0}$ w.r.t. $\omega$ depend only on $e_{\omega}$ and $e_{\eta, \omega}$ which are such that:

$$1 + (1 - \rho)e_{\omega} = (\gamma - \rho)e_{\eta, \omega}. \quad (34)$$

This relationship in particular implies that within any regime such that there is no qualification $s$ both covered by the minimum wage and such that $\bar{\pi}_{t,0} = \left(\frac{\omega}{\phi_t}\right)^{\delta}$ then $e_{\eta, \omega} = 0$ and $e_{\omega} = -\sigma$, since $S_\omega = 0$. That is for all values of $\omega$ within such a regime, the quantities $\bar{\pi}_{t,0}$, any such qualification is not covered by the minimum wage, are constant.

This can take place for example whenever $\omega$ is greater than $\frac{\sigma}{\delta + \gamma}$ the competitive wage for the $Q$ qualification (see proposition 4.). This is however not the only case as Proposition 3 implies, such a situation can arise whenever the differentials between the competitive wage of the occupations covered by the minimum wage and the occupations not covered are large enough, i.e. greater than $\frac{1}{\delta + \gamma}$. We observe that (24) implies that $S_\omega(\omega) \geq 0$ and therefore we deduce $e_{\omega} \geq -\sigma$ and because of (34) we have $e_{\eta, \omega} \geq 0$. It is however possible for $e_{\omega}$ to be positive for example if the increase in the minimum wage implies an increase in the demand smaller than that of the supply for the qualifications covered by the minimum wage.

$L$ is defined as

$$(f(\omega))^{1/\nu} \equiv L \left(1 + \sum_{k=1}^{Q} \bar{\pi}_{k,0}\right),$$

Hence a change in the minimum wage must lead to variation in the labour aggregate $L$, such that:

$$\frac{f(\omega)^{1/\nu}}{\omega} \sum_{k=1}^{Q} \alpha_{k,0} \bar{\pi}_{k,0}^\nu e_{k,0, \omega} = \frac{dL}{d\omega} \left(1 + \sum_{k=1}^{Q} \bar{\pi}_{k,0}\right) + \frac{L}{\omega} \sum_{k=1}^{Q} \bar{\pi}_{k,0} e_{k,0, \omega}$$

$$\Leftrightarrow \sum_{k=1}^{Q} \frac{\alpha_{k,0} \bar{\pi}_{k,0}}{f(\omega)} e_{k,0, \omega} = e_{\omega} + \frac{1}{1 + \sum_{k=1}^{Q} \bar{\pi}_{k,0}} \sum_{k=1}^{Q} \bar{\pi}_{k,0} e_{k,0, \omega},$$
which gives:
\[
e_4^{\omega} = \sum_{k=1}^{q} e_{k,\theta_k} \left( \frac{\alpha_{k,0} \pi_{k,0}^{\rho}}{f_2(\omega)} - \frac{\pi_{k,0}}{1 + \sum_{k=1}^{q} \pi_{k,0}} \right),
\]
and the second expression announced in the proposition can be deduced from earlier definitions.

**Proof of Proposition 6**

The first order conditions (30), for the 1st and 3rd qualification imply that:
\[
\tilde{\pi}_{1,0}(\omega) = \left( \frac{\alpha_1}{\phi_1} \right)^{\sigma_{1,0}} \tilde{\Lambda}(\omega) = \pi_{1,0}^{\star} \tilde{\Lambda}(\omega) \quad \text{and} \quad \pi_{3,0}(\omega) = \left( \frac{\alpha_3}{\phi_3} \right)^{\sigma_{3,0}} \tilde{\Lambda}(\omega) = \pi_{3,0}^{\star} \tilde{\Lambda}(\omega),
\]
and the optimality conditions for the 2nd qualification state that $\omega$ is such that:
\[
\frac{\omega \pi_{1,0}}{C_{-1}(\omega)} < \frac{\omega \pi_{2,0}}{f_{-1}(\omega)} \leq \frac{\omega \pi_{2,0}}{C_{-1}(\omega)},
\]
with
\[
C_{-1}(\omega) \equiv 1 + \phi_1 \left( \pi_{1,0}(\omega) \right)^{\gamma} + \sum_{k=2}^{3} \pi_{k,0}(\omega) \max \left\{ \omega, \phi_k \left( \pi_{k,0}(\omega) \right)^{\gamma} \right\},
\]
\[
f_{-1}(\omega) \equiv 1 + \sum_{k=1}^{3} \alpha_{k,0} \pi_{k,0}^{\rho},
\]
and we have $\gamma \tilde{\Lambda}(\omega)^{\gamma-\rho} = \frac{C_{-1}(\omega)}{f_{-1}(\omega)}$.

Let $A_{-1} = \alpha_1 \pi_{1,0}^{\rho} + \alpha_3 \pi_{3,0}^{\rho} = a_{1,0}^{\rho} + \phi_3 \pi_{3,0}^{\gamma}$. The first order conditions determine $\tilde{\Lambda}(\omega)$ uniquely such that:
\[
\frac{\tilde{\Lambda}^{\rho}}{1 + A_{-1} \tilde{\Lambda}^{\rho} + \alpha_2 \pi_{2,0}^{\rho}} = \gamma \frac{\tilde{\Lambda}^{\gamma}}{1 + A_{-1} \tilde{\Lambda}^{\gamma} + \phi_2 \pi_{2,0}^{\gamma}},
\]
\[
\Leftrightarrow \gamma \left( 1 + \alpha_2 \pi_{2,0}^{\rho} \right) \tilde{\Lambda}^{\gamma-\rho} + (\gamma - 1) A_{-1} \tilde{\Lambda}^\gamma = 1 + \phi_2 \pi_{2,0}^{\gamma}.
\]
We note that as function of $\lambda$ over $[0,1]$ the LHS is increasing from $0$ to $\gamma (1 + \alpha_2 \pi_{2,0}^{\rho}) + (\gamma - 1) A_{-1} > 1$, while the RHS is constant. Hence the solution $\tilde{\Lambda}$ is strictly less than 1.

Given the structure of the solution, we can use this expression to determine the effect of $\omega$ on $\tilde{\Lambda}(\omega)$, we have:
\[
\frac{\omega}{\Lambda} \frac{d\tilde{\Lambda}}{d\omega} \left\{ (\gamma - \rho) \left( 1 + \alpha_2 \pi_{2,0}^{\rho} \right) \tilde{\Lambda}^{\gamma-\rho} + (\gamma - 1) A_{-1} \tilde{\Lambda}^\gamma \right\} = \delta \left\{ \phi_2 \pi_{2,0}^{\gamma} - \rho \alpha_2 \pi_{2,0}^{\rho} \tilde{\Lambda}^{\gamma-\rho} \right\}.
\]
The term within brackets on the LHS is always positive. Observe that $\phi_2 \pi_{2,0}^{\rho} = \omega \pi_{2,0}^{\rho}$, and the optimality condition for the 2nd qualification demands $\omega \pi_{2,0}^{\rho} < \alpha_2 \pi_{2,0}^{\rho} \tilde{\Lambda}^{\gamma-\rho} \leq \gamma \omega \pi_{2,0}^{\rho}$, hence the RHS is always positive since $\rho < 1$. Some further algebra shows that $\frac{\omega}{\Lambda} \frac{d\tilde{\Lambda}}{d\omega} < \delta$. 

-37-
To prove $\pi_{1,0}(\omega) \leq \left( \frac{\omega}{1 + \phi_1} \right)^\phi$, assume instead that $\pi_{1,0}(\omega) > \left( \frac{\omega}{1 + \phi_1} \right)^\phi$. In that case the first order condition states that

$$\alpha_1 \frac{\bar{C}_1(\omega)}{\bar{F}_1(\omega)} = \gamma \phi_1 \pi_{1,0} \gamma^{-p}$$

And this would imply

$$\alpha_1 \frac{\bar{C}_1(\omega)}{\bar{F}_1(\omega)} > \gamma \phi_1 \left( \frac{\omega}{1 + \phi_1} \right)^\phi \Rightarrow \phi_1 \left( \frac{\alpha_1}{1 + \phi_1} \right)^1 \left( \frac{1}{\gamma \bar{F}_1(\omega)} \right)^1 > \omega^\phi$$

Some simplifications lead to:

$$\Rightarrow w_{1,0}^* \bar{A}(\omega) > \omega^\phi \Rightarrow w_{1,0}^* > \omega^\phi,$$

which would in turn mean that $\omega < w_{1,0}^* < w_{2,0}^*$, where the last inequality arises from our initial assumption that the 2nd occupation is paid better than the first in the competitive equilibrium. Finally if $\omega < w_{1,0}^* < w_{2,0}^*$, the minimum wage cannot affect the market for the second qualification which contradicts our earlier assumption. Hence, $\pi_{1,0}(\omega)$ cannot be larger than $\left( \frac{\omega}{1 + \phi_1} \right)^\phi$, and since the first qualification is on the supply curve this means workers with that qualification are paid less than $\omega$.■
Online Appendix: The Effect of a Minimum Wage in Competitive Markets

When a minimum wage is introduced, several effects come into play. First because the minimum involves an increase in the marginal cost of production the scale of production may vary, this is the usual scale effect. Second the minimum wage will affect some qualifications and not others. This will generate a redistribution of the demand and the supply of qualifications because the relative rewards and costs are modified. The new equilibrium set of wages must accounts for both effects.

The introduction of the minimum wage in this simple model changes the scale of operations, it changes the relative marginal costs to the firm of workers with distinct qualifications and it changes the distribution of the supply of qualification.

In the presence of a minimum wage the calculus of the equilibrium wages and proportions is modified significantly since equating the proportions demanded and supplied of each qualification does not account for the possibility of some unemployment created by a mismatch between the demand side and the supply side.

Instead, the market equilibrium arises when the quantity of labour supplied, i.e. population size, say $M$, times the proportions in the population choosing each qualification, matches the quantity demanded, now the total amount of labour demanded times the proportions in the population of workers demanded, given the wage for each qualification such that the minimum wage does not bite. For qualification where the minimum wage bites the supply side is determined relative to the wage of the $0^{th}$ qualification, while the demand side is determined by the same relative wage in equilibrium and the scale of production overall.

That is we have an equilibrium in the labour markets where the minimum wage $w$ is imposed when $w^* = (w_0^*, w_1^*, ..., w_q^*, ..., w_Q^*)$ is such that

$$M \pi_q^s(w^*) \geq N^*(w^*) \pi_q^d(w^*)$$

for all $q=0...Q$. \hspace{1cm} (A1)

and such that

$$M \pi_q^s(w^*) = N^*(w^*) \pi_q^d(w^*)$$

(A2)

for all qualification $q$ where the minimum wage does not bite, i.e. such that $w_q^* > w$.

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We now describe this equilibrium in details.

To complete the model in a simplified way, we assume that
\[ \psi(x) = x^\alpha \] with \( 0 < \alpha \leq 1 \).

We maintain the assumption that the 0\textsuperscript{th} qualification will command a wage larger than the minimum wage in the new equilibrium. The first order conditions above imply that the demand for new labour market entrants with qualification \( q \) is:
\[ N_q^* = N_0 \theta_q \left( \frac{w_q}{w_0} \right)^{-\sigma} \]
for all \( q = 1, \ldots, Q \) \( (A4) \)

given the wages \( w_0, \ldots, w_q, \ldots, w_Q \), and \( N_0 \), with \( \theta_q = \alpha_q^{\sigma} \).

This in turn implies that, given the wages \( w_0, \ldots, w_q, \ldots, w_Q \), the total cost of employing \( N_0, N_1^*, \ldots, N_Q^* \) is:
\[ \sum_{q=0}^{Q} w_q N_q = w_0 N_0 + \sum_{q=1}^{Q} w_q N_0 \theta_q \left( \frac{w_q}{w_0} \right)^{-\sigma} \]
\[ = N_0 \left[ w_0 + \sum_{q=1}^{Q} w_q \theta_q \left( \frac{w_q}{w_0} \right)^{-\sigma} \right] = N_0 \Omega(w), \]
where we can think of \( \Omega(w) \) as the marginal cost of employing labour market entrants with the 0\textsuperscript{th} qualification. Clearly this marginal cost depends on the cost of all qualifications (i.e. the wages). It is increasing in \( w_0 \) and depending on the value of \( \sigma \) it is increasing \((\sigma < 1)\) or decreasing \((\sigma > 1)\) with the cost of any other qualification.

The aggregate employment in production becomes:
\[ F(N_0, N_1^*, \ldots, N_Q^*) = N_0 \left[ \alpha_0 + \sum_{q=1}^{Q} \alpha_q \theta_q \left( \frac{w_q}{w_0} \right)^{-\sigma} \right]^{-\frac{1}{\alpha}} = N_0 \Theta(w). \]

\( (A6) \)

The aggregate is homogenous of degree 0 in the wages, i.e. only the relative wages matter here. Assuming that the economy equalises the marginal revenue to the marginal cost, the demand for the 0\textsuperscript{th} qualification satisfies:
\[ p \alpha \Theta(w)^\alpha N_0^{\alpha-1} = \Omega(w) \Rightarrow N_0^*(w) = \left( \frac{\Omega(w)}{p \alpha \Theta(w)^\alpha} \right)^{\frac{1}{\alpha-1}}. \]

\( (A7) \)

After simplifications we obtain:
\[ N^*_0(w) = \left( \frac{w_0}{p} \right)^{1-\alpha} \left[ 1 + \sum_{q=1}^{Q} \theta_{q,0} \left( \frac{w_q}{w_0} \right)^{-1-\sigma} \right]^{-\frac{1}{\alpha-1}} \left( \alpha \alpha_0^{\alpha p} \right)^{\frac{1}{1-\alpha}}. \]

Note that the demand is homogenous of degree zeros in the product price \( p \) and in the wages. We show in the appendix that the demand curve for the \( o^{th} \) qualification is decreasing and approaches 0 as \( w_0 \) tends to infinity. If the labour markets for the other qualifications are in equilibrium, such that the wages relative to \( w_0 \), \( w_q^* \), remain fixed, the elasticity of the labour demand for the \( o^{th} \) qualification is equal to \( \frac{1}{\alpha-1} \), otherwise in general the elasticity of labour demand can be smaller or larger depending on whether or not \( -\sigma \geq \frac{1}{\alpha-1} \).

Finally, the demand for total employment in the economy can be expressed as:

\[ N^* = \sum_{q=0}^{Q} N_q = \left[ 1 + \sum_{q=1}^{Q} \theta_{q,0} \left( \frac{w_q}{w_0} \right)^{-\sigma} \right]^{-\frac{1}{\alpha-1}} \left( \alpha \alpha_0^{\alpha p} \right)^{\frac{1}{1-\alpha}} N^*_0(w), \tag{A8} \]

and we note that this agrees with our earlier findings about the relative demands for each qualification and in particular qualification 0.

To obtain the equilibrium wages and supply and demand proportions in each qualification, we propose the following approach (inspired from the conventional treatment of minimum wages in a simple partial equilibrium supply and demand model):

i) Calculate the competitive relative wages as described in the earlier section.

ii) Determine the supply and demand schedule for the \( o^{th} \) qualification as a function of the wage for that qualification, the minimum wage and the competitive relative wages on the market for qualifications unaffected by the minimum wage.

iii) Determine the equilibrium wage for the \( o^{th} \) qualification.

iv) Derive the absolute equilibrium wages and the equilibrium quantities.

The analysis shows the following:

i) Assuming that the markets the \( Q \) other qualifications, \( q=1...Q \), are in equilibrium, the demand curve for the \( o^{th} \) occupation is decreasing with \( w_0 \) and tends to 0 as \( w_0 \) goes to infinity. These properties apply whether or not a minimum wage applies to one or more qualifications.

ii) Without a minimum wage the proportion of the population supplying the \( o^{th} \) occupation is positive and constant (i.e. it does not depend on \( w_0 \)) if all other labour markets are in equilibrium. The equilibrium wage and quantities on the market for the \( o^{th} \) qualification are
\[ N_0^*\left( w_0^* \right) = \frac{M}{1 + \sum_{q=1}^{Q} \exp(\delta_q) \left( w_{q,0}^* \right)^\delta}, \quad \text{(A9)} \]

\[ w_0^* = \frac{\alpha p \alpha_0^{(\rho)} \rho}{M^{1-\alpha}} \left[ 1 + \sum_{q=1}^{Q} \exp(\delta_q) \left( w_{q,0}^* \right)^\delta \right]^{1-\alpha} \left[ 1 + \sum_{q=1}^{Q} \theta_{q,0} \left( w_{q,0}^* \right)^{1-\sigma} \right]^{-\frac{\alpha}{\sigma-1}} \quad \text{(A10)} \]

and the equilibrium wages on the other Q markets can be obtained directly with

\[ w_q^* = w_0^* w_{q,0}^* = w_0^* \exp \left( \frac{\ln \theta_{q,0} - \delta_q}{\delta + \sigma} \right) \quad \text{for all } q = 1, \ldots, Q. \quad \text{(A11)} \]

The equilibrium proportions of the workforce in each qualification remain as described in (A4).

Points i) and ii) together show that, provided all the other markets are in equilibrium, the demand curve \( N_0^*\left( w_0 \right) \) crosses the supply curve once only. This implies that for a wage \( w_0 \) smaller than \( w_0^* \) the demand curve is above the supply curve while for a wage larger than the equilibrium wage the demand curve is below the supply curve.

iii) When a minimum wage, \( w' \), is introduced the proportion of the population supplying the \( 0^\text{th} \) occupation is positive, less than or equal to the competitive proportion and increases with \( w_0 \). For \( w_0 \) large (i.e. such that the minimum wage does not bite) the proportion supplied is equal to the competitive quantity in the absence of a minimum wage. Analytical expressions for the equilibrium wage, \( w_0^* \), and quantity, \( N_0^*\left( w_0^* \right) \) on the market for the \( 0^\text{th} \) qualification are no longer available in general. Numerical evaluation is relatively simple however.

Hence we conclude that, for a reasonable value of the minimum wage, an equilibrium always exists on the market for the \( 0^\text{th} \) qualification where the demand meets the supply at a positive wage and for positive quantities (although if the minimum wage is too large the equilibrium would fail to exist). In the absence of a minimum wage the equilibrium wage \( w_0^* \) and the relative wages \( w_{q,0}^* \) for all \( q=1\ldots,Q \) describe the equilibrium on the Q+1 labour markets.
Figure 2:

In the presence of a minimum wage, $w$, both the demand and the supply curve for the $0^{th}$ occupation are modified. However, a unique equilibrium as described by (14) and (15) always exists. That is if the minimum wage bites for one or more qualification, the supply for this qualification exceeds the demand, in all other markets the supply meets the demand.

Since the supply curve is (weakly) increasing with $w$, introducing a minimum wage always leads to a reduction of the equilibrium number of workers employed with the $0^{th}$ occupation. However the effect of the minimum wage on the equilibrium wage for the $0^{th}$ occupation depends on the scale of production and the elasticity of substitution. Figure 1 illustrates our finding. It shows the relative positions of the equilibrium wages $w_0^*$ and $w_0^{**}$, before and after the introduction of the minimum wage. The vertical dotted lines indicate the values of $w_0$ such that the minimum wage bites on a further qualification market, i.e. $w(w_{q,0}^{*})^{-1}$, $q=1...Q$. In the example shown, $Q=4$ and the minimum wage is set to bite on the first qualification only. Note that in the presence of a minimum wage the demand curve for the $0^{th}$ qualification, given the equilibrium on the markets for the other qualifications, is decreasing but no longer convex.
The Incidence Of The Minimum Wage On The Competitive Wage Distribution

We characterise the effect of the introduction of a minimum wage on the equilibrium wage on the market of the 1\textsuperscript{st} qualification (the least paid qualification). We determine the equilibrium wage $w^*_0$ based on simple Taylor expansions of the supply and demand schedules on the market for the 0\textsuperscript{th} qualification near the relative wage $w^*_{1,0}$ such that for a small positive $\varepsilon$,

$$\ln \frac{w}{w_1} = \varepsilon.$$  Hence our analysis holds, in the first instance, on the first branch of the demand and supply curves when the minimum wage bites on the market for the 1\textsuperscript{st} qualification only.

For a minimum wage such that, $\ln \frac{w}{w_1} = \varepsilon$ with $\varepsilon$ small and positive, $w^*_0$ satisfies:

$$\left(\frac{w^*_0}{w_0}\right)^{1-\frac{1}{\alpha}} \left\{ \theta_{1,0} \left( \frac{w}{w_0} \right)^{1-\frac{1}{\alpha}} + 1 + \sum_{q=2}^{Q} \theta_{q,0} \left( w^*_{q,0} \right)^{1-\frac{1}{\alpha}} \right\}^\kappa \left(1 + \frac{1}{\exp\left(\frac{w}{w_0}\right)^{\frac{1}{\rho}}} + 1 + \sum_{q=1}^{Q} \exp\left(\delta_q\right) \left( w^*_{q,0} \right)^{\gamma} \right)^{\delta_2}.$$

with $\kappa \equiv \frac{1}{\alpha - 1} \left( 1 - \frac{\alpha}{\rho} \right)$ and $\Gamma = \left( \alpha \rho \theta_{1,0}^{\alpha/\rho} \right)^{1-\frac{1}{\alpha}}$. The quantities $D_2$ and $S_2$ remain constant for a small enough increase in the minimum wage, i.e. the markets for all other qualifications clear at their competitive relative wages, $w^*_{q,0}$ for $q = 2..Q$. Taking logarithms on each side gives:

$$\frac{1}{\alpha - 1} \ln w^*_0 + \kappa \ln \left( \theta_{1,0} \left( \frac{w}{w_0} \right)^{1-\frac{1}{\alpha}} + D_2 \right) + \ln \left( \exp\left(\delta_1\right) \left( \frac{w}{w_0} \right)^{\gamma} + S_2 \right) + \ln \Gamma = 0$$

and expanding the second and third term to the first order around $\ln \frac{w^*_1}{w_0}$, gives
\[-\ln \left( \pi_0^D \left( w_0^*, w, w_{2a}^* \right) \right) \equiv \ln \left( \frac{\theta_1.0 \left( \frac{w}{w_0^*} \right)}{1-\sigma} + D_2 \right) \]

\[\approx -\ln \left( \pi_0^D \left( w^* \right) \right) + (1-\sigma) \pi_1^D \left( w^* \right) \left( \ln \frac{w}{w_0^*} - \ln w_{1,0}^* \right)\]

\[-\ln \left( \pi_0^D \left( w_0^*, w, w_{2a}^* \right) \right) \equiv \ln \left( \exp(\delta) \exp \left( \delta \ln \frac{w}{w_0^*} \right) + S_2 \right) \]

\[\approx -\ln \left( \pi_0^D \left( w^* \right) \right) + \delta \pi_1^D \left( w^* \right) \left( \ln \frac{w}{w_0^*} - \ln w_{1,0}^* \right)\]

around \( w_{1,0}^* \) the intercepts of both expansions are equal to each other, that is at \( w_{1,0}^* \) the market for the \( 0^{th} \) qualification would be in equilibrium without a minimum wage and equal to \(-\ln \pi_0^*\), the proportion of all employees with the \( 0^{th} \) qualification. Substituting the expressions above into the equilibrium condition gives:

\[
\frac{1}{\alpha-1} \ln w_0^* + \pi_1^* \left( -\ln \pi_0^* + (1-\sigma) \pi_1^* \left( \ln \frac{w}{w_0^*} + \ln w_{1,0}^* \right) \right)
\]

\[-\ln \pi_0^* + \delta \pi_1^* \left( \ln \frac{w}{w_0^*} - \ln w_{1,0}^* + \ln w_0^* \right) + \ln \Gamma = 0,\]

and after some manipulations we obtain:

\[
\left\{ \frac{1}{1-\alpha} + \pi_1^* \left( \kappa (1-\sigma) + \delta \right) \right\} \ln w_0^* =
\]

\[-(\kappa+1) \ln \pi_0^* + \ln \Gamma + \pi_1^* \left( \kappa (1-\sigma) + \delta \right) \ln w_0^* + \pi_1^* \left( \kappa (1-\sigma) + \delta \right) \ln \frac{w}{w_0^*},\]

which eventually simplifies to:

\[
\ln w_0^* = \ln w_0^* + \pi_1^* \frac{1-\pi_0^*}{\kappa (1-\sigma) + \delta} \ln \frac{w}{w_0^*},
\]

\[
\equiv \ln w_0^* + \gamma^* \ln \frac{w}{w_0^*},
\]

with

\[
\gamma^* = \pi_1^* \frac{1}{\alpha-1} + \sigma + \delta, \quad \text{and} \quad \gamma^* \leq 1.
\]
Since the denominator in (A12) is always positive, the sign of the effect of the minimum wage on the equilibrium wage of the $0^{th}$ qualification is determined by the numerator only. Hence the effect of the introduction of a minimum wage on the market for the $1^{st}$ qualification reduces the equilibrium wage of the more valuable qualification if $\frac{1}{\alpha - 1} + \sigma + \delta$ is negative, that is whenever

$$\frac{1}{1 - \alpha} > \sigma + \delta \text{ or equivalently } \frac{1}{\sigma + \delta + \alpha} > 1.$$ 

Under this condition, the minimum wage narrows (reduces the width of the support of) the distribution of wages in the economy, since it increases the smaller wage and decreases the wage of the better paid qualification.

The analysis of the relative position of the demand curves with and without a minimum wage concludes that, if $\rho > \alpha$, the demand for the $0^{th}$ qualification is always above (to the right of) the demand in the absence of a minimum wage and as a consequence the equilibrium wage is larger than it would be without it. In this case the minimum wage increases the average wage throughout the economy, that is not only the least paid qualifications benefit from a higher wage because of the minimum wage, but since the incidence of the minimum wage on the better paid labour market is positive, i.e. $w_0^{**} \geq w_0^*$, all other qualifications benefit as well. Note that in these markets the wages relative to the equilibrium wage $w_0^{**}$ remain constant at their competitive levels. Hence it is possible for the average wage in the economy to increase while the variance of the relative wages decreases.

Alternatively, if $\frac{1}{1 - \alpha} < \delta + \sigma$, the equilibrium wage $w_0^{**}$ is larger than $w_0^*$ although the relative increase is always strictly less than $\ln \frac{w}{w^*_i} = \varepsilon$, since $\gamma^* \leq 1$.

In other words if $\frac{1}{1 - \alpha} < \delta + \sigma$, the introduction of the minimum wage does not lead to the shifting up of the wage distribution by $\frac{w}{w^*_i}$.

We characterise the effect of changes in the parameters $\alpha$, $\sigma$ and $\delta$ on the incidence parameter $\gamma^*$:

- $\frac{\partial \gamma^*}{\partial \alpha} = -\frac{\sigma + \delta}{(\alpha - 1)^2} \frac{1}{\Delta} < 0$, with $\Delta \equiv \frac{1 - \pi^*_i}{1 - \alpha} + (\sigma + \delta)\pi^*_i$, that is the larger the return to scale parameter the smaller the incidence of the minimum wage on the wage of the $0^{th}$ qualification. Since the equilibrium proportion, $\pi^*_i$, in the absence of a minimum wage does not depend on $\alpha$, the effect of $\alpha$ on $\gamma^*$ is unambiguously negative.
\[
\frac{\partial \gamma^*}{\partial \sigma} = \pi_i^* (1 - \gamma^*) + \frac{1}{\Delta} \frac{\delta}{\delta + \sigma^2} \gamma^* \left\{ \mu_i - \bar{\mu} \right\}.
\]

The second term arises because of the effect of \( \sigma \) on the equilibrium proportion \( \pi_i^* \), i.e. as \( \sigma \) changes it changes the equilibrium proportion without a minimum wage. \( \frac{\partial \gamma^*}{\partial \sigma} \) is positive if and only if

\[
\frac{(\delta + \sigma^2)^2}{\delta} \geq \left\{ \frac{1}{1 - \alpha} - \sigma - \delta \right\} \left\{ \mu_i - \bar{\mu} \right\}.
\]

The inequality will be satisfied if the two terms on the RHS have opposite signs, which the model allows without restrictions. If we expect the minimum wage to have a negative incidence on \( w_{0}^* \), the first term is positive. The second term depends on the distribution of the “constant” terms in the utility and of the relative productivity parameters in the labour input aggregate and can take any sign, and hence for the effect of the elasticity of substitution on the incidence to be positive we would require \( \mu_i \geq \bar{\mu} \).

Similarly

\[
\frac{\partial \gamma^*}{\partial \delta} = \pi_i^* (1 - \gamma^*) + \gamma^* \frac{1}{\Delta} \frac{\sigma}{1 - \alpha \left( \delta + \sigma^2 \right)} \left\{ \mu_i - \bar{\nu} \right\}
\]

which is positive if and only if

\[
\frac{(\delta + \sigma^2)^2}{\sigma} \geq \left\{ \frac{1}{1 - \alpha} - \sigma - \delta \right\} \left\{ \mu_i - \bar{\nu} \right\}.
\]

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The overall effect of \( w \) on the demand for the 0th qualification is however always negative. Indeed the previous analysis carries further to the sign of the elasticity of \( N_0^* \) with respect to \( w \) when it bites only on the market for the first qualification and when its increase relative to the competitive wage \( w_1^* \) is small.

We have:

\[
\frac{d \ln \left( N_0 \left(w_0^{**}, w, w_2^{*}, w, w_2^{*}\right) \right)}{d \ln w} = \frac{d \ln N_0^{**}}{d \ln w_0^{**}} \frac{d \ln w^{**}}{d \ln w} + \frac{d \ln N_0^{**}}{d \ln w}
\]

\[
= \gamma^* \left( \frac{1}{\alpha - 1} \gamma^* + \kappa(\sigma - 1) \pi_i^* \right) + \kappa(1 - \sigma) \pi_i^*
\]

\[
= \gamma^* \left( \pi_i^* \delta - \frac{1}{1 - \alpha} \pi_i^* (\sigma + \delta) \right) + \kappa(1 - \sigma) \pi_i^*
\]

\[
= \pi_i^* \delta (\gamma^* - 1) \leq 0
\]

which is always negative since \( \gamma^* \leq 1 \). Since the equilibrium quantity of workers with the 0th qualification falls when the minimum wage is introduced, the equilibrium quantities for all other qualification will fall as well since the conditions (A4) must be satisfied and the relative wages in equilibrium on these markets are fixed at their competitive levels \( w_{q,0}^{**} \) for \( q = 2, \ldots, Q \).
We deduce further the effect of the minimum wage on the wage bills for the 0th qualification (i.e. the effect of the minimum wage on the amount firm spend on the better rewarded qualification). We consider the elasticity of the wage bill with respect to the minimum wage when it only applies to the first qualification ($q=1$). Some algebra further implies:

\[
\frac{d \ln \left( N_0 \left( w_0^*, \left( w, v_{2q} \right), \left( w, v_{2q} \right) \right) \right)}{d \ln w} = \frac{d \ln w_0^*}{d \ln w} \left( \frac{d \ln N_0^*}{d \ln w_0^*} + 1 \right) + \frac{d \ln N_0^*}{d \ln w}
\]

\[
= \gamma^* \left( \frac{1}{\alpha - 1} + \kappa (\sigma - 1) \pi^*_1 + 1 \right) + \kappa (1 - \sigma) \pi^*_1
\]

\[
= \gamma^* \left( 1 + \pi^*_1 \delta - \frac{\pi^*_1}{1 - \alpha} - \pi^*_1 (\sigma + \delta) \right) + \kappa (1 - \sigma) \pi^*_1
\]

\[
= \gamma^* (1 + \pi^*_1 \delta) - \pi^*_1 \delta.
\]

This last expression is positive if and only if $\gamma^* \geq \frac{\delta \pi^*_1}{1 + \delta \pi^*_1} \geq 0$ or if and only if

\[
\frac{\delta \pi^*_1}{1 - \gamma^*} \Leftrightarrow \frac{\alpha \delta + 1}{1 - \alpha} \leq \sigma. \]

Hence if the elasticity of substitution is large enough the introduction of the minimum wage can potentially increase the wage bill for the 0th qualification.