The Marginal Rate of Substitution and the Specification of Labour Supply Models

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Abstract

In this note I revisit Heckman's proposal, [Heckman, 1974], to specify a static labour supply model using a simple formulation for the Marginal Rate of Substitution between total expenditure on consumption and hours of work. I describe the analytical form of the expenditure function and I show how the direct and indirect utility functions can be recovered. I propose an alternative specification for the MRS and in this case I describe analytically the labour supply functions, the indirect and direct utility functions as well as the expenditure function.

Keywords: Labour Supply Model; Specification; Marginal Rate of Substitution.

Introduction

The question of the appropriate specification of static labour supply model has received much attention. The literature has proposed several specifications such that the Marshallian labour supply function and the utility function, either in its direct or its indirect form, are known analytically. Stern [1986] provides a detailed discussion of many potential specifications. Blundell and MaCurdy [1999] compare and contrast several specification forms

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found in the empirical literature, and Blundell et al. [2007] discuss the question of estimation of labour supply models and the implications of the specification choices on the empirical practice.
The Marshallian labour supply function is usually specified directly and the literature focuses on specifications which involve three distinct parameters to capture (i) the dependence of the labour supply level on demographic characteristics, (ii) the response of the labour supply to the wage rate, and (iii) the response to the level of unearned income. For example, the linear labour supply function belongs to this class, and in that case the direct and indirect utility functions and the expenditure function are known in a closed form [Stern, 1986] . These are useful properties to analyse the welfare consequences of tax reforms for example, and to calculate the likelihood in the presence of fixed costs to work, or more generally in the presence of a non concave budget set, see van Soest et al. [1993]; Bloemen and Kapteyn [2008]. The number of specifications of the labour supply function with an analytical form for the utility function (direct or indirect) is small however.

In an early paper on labour supply and child care, Heckman [1974] specifies the shape of the preferences he estimates by describing instead the functional form for the marginal rate of substitution between the hours of work and the expenditure on consumption goods, $\omega(h, c)$. Such a specification choice is not common in the literature, and Stern [1986], for instance, does not review Heckman's proposal.

The specification Heckman proposes depends on three parameters but defines the labour supply function implicitly only. Because the marginal rate of substitution is directly related to the shape of the indifference curves it is however easy to impose the theoretical restrictions required for the quasi-concavity of the utility function, whereas the same requirements demand more effort in a direct specification of the labour supply function, see Stern [1986]. Heckman does not state either the form of the direct nor the form of the indirect utility function, and perhaps as a consequence Heckman's specification choice or his general approach of specifying a marginal rate of substitution is not common in the literature.

In this note I characterise the expenditure function that is consistent with Heckman's specification and I show how to recover the direct and indirect utility functions. I show furthermore that a closely related alternative specification of the marginal rate of substitution leads to analytically closed forms for the Marshallian and Hicksian labour supply function and for both the direct and the indirect utility functions. Altogether, these provides further alternatives to existing formulations that may be useful in practice.

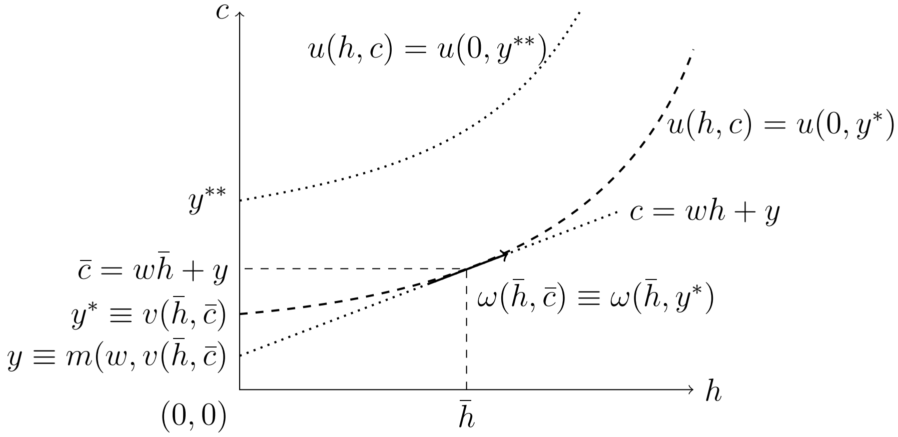
Heckman's specification

Given some utility function $u(h, c)$, Heckman specifies the marginal rate of substitution, $\omega(h, c) \equiv -\frac{u_h(h,c)}{u_c(h,c)}$, at any point (h, c) such that it depends on h, the hours of work, and on y^* the level of unearned income consistent with zero hours on the indifference curve which passes through (h, c). In particular he assumes:

$$\ln \omega(h, c) \equiv \ln \omega(h, y^*) = \alpha + \beta h + \gamma y^*. \tag{1}$$

Figure 1 illustrates the intuition: given a level y^*, we expect the marginal rate of substitution to be increasing with the level of hours, h (assuming $0 \leq h \leq T$, where T is the total time endowment). If we assume that

Figure 1: Preferences on Expenditure and Hours of Labour



leisure is a normal good we expect further that $\omega(h, y^*)$ increases with y^* (the indifference curves would tilt to the left as y^* increases, see Bilancini and Boncinelli [2010]), hence we expect $\beta > 0$ and $\gamma > 0$. These are the conditions Heckman imposes on the parameters.

From this initial point, Heckman shows that the optimal labour supply, $\bar{h}$, for a given wage w and for a level of unearned income y solves the equation:

$$w\bar{h} + y = \int_0^{\bar{h}} \omega(\eta, y^*(\bar{h}, w))d\eta + y^*(\bar{h}, w) \tag{2}$$

where $y^*(\bar{h}, w)$ is given by $w = \omega(\bar{h}, y^*(\bar{h}, w))$. Applied to the specification of the marginal rate of substitution given in (1), the optimal labour supply is then implicitly defined as the solution $\bar{h}(w, y)$ of the equation:

$$\ln w = \alpha + \beta h + \gamma\big(wh + y - \frac{w}{\beta}(1 - \exp(-\beta h))\big). \tag{3}$$

Deriving the utility and expenditure functions

To start the analysis observe that $u(0, y^*)$ is a particular cardinalisation of the utility function, and that y^* is another such version of the utility function. Hence for all labour supply and expenditure plans (h, c) such that $u(h, c) =$

$u(0, y^*)$ we have $y^* = g(u(h, c))$ where the function $g(x)$ is increasing with x. Call $v(h, c) \equiv g(u(h, c))$. (1) requires that

$$-\frac{u_h(h,c)}{u_c(h,c)}|_{u(h,c)=u(0,y^*)} = -\frac{v_h(h,c)}{v_c(h,c)}|_{v(h,c)=y^*} = \exp(\alpha + \beta h + \gamma y^*),$$

which we can rewrite as a first order homogeneous partial differential equation (omitting the arguments of the utility function to clarify the expression):

$$v_h + \exp(\alpha + \beta h + \gamma y^*) v_c = 0, \tag{4}$$

and we require that $v(h, c) = y^* \ \forall h \in [0, T]$.

The simplest and most direct method to solve this type of partial differential equation problem (a Cauchy problem, see Marsden et al. [2005]), uses the method of characteristics. In our problem a particular characteristic corresponds to a single indifference curve, $c(h, v(0, y^*)) \ \forall h \in [0, T]$. Indeed we can verify that along an indifference curve at the level $v(0, y^*)$, the original specification implies:

$$\frac{dc(h, v(0, y^*))}{dh} = \exp(\alpha + \beta h + \gamma y^*) \tag{5}$$

with the additional requirement for the solution that $c(0, v(0, y^*)) = y^*$. Hence the equation for an indifference curve, i.e. c as a function of h for a given value of y^*, takes the form:

$$c(h, v(0, y^*)) = \frac{\exp(\alpha + \gamma y^*)}{\beta}\big(\exp(\beta h) - 1\big) + y^*, \tag{6}$$

and we verify that $c(0, v(0, y^*)) = y^*$. Note that this step of the process corresponds exactly to the reasoning that leads Heckman to equation (2).

Finally given h, c, solving (6) for y^* delivers an implicit representation of the direct utility function $v(h, c)$. For any given value of (h, c) it is straightforward to show that the solution $y^*(h, c)$ is unique since the left hand side is strictly increasing between $-\infty$ and $+\infty$ as a function of y^*.

The initial specification of the marginal rate of substitution can be alternatively seen as a specification of the compensated (Hicksian) labour supply function:

$$h(w,v) = \frac{\ln w - \alpha - \gamma v}{\beta}, \tag{7}$$

which requires that $\ln w > \alpha+\gamma v$ for a positive labour supply. If the condition is not satisfied then the optimal labour supply is $h(w,v) = 0$ and the level of expenditure required to reach utility v is $m(w,v) = y^* = v$. If instead $\ln w > \alpha + \gamma v$, substituting (7) in (6) and using the budget constraint, leads to an expression for the expenditure function:

$$m(w,v) = \frac{w}{\beta}\big(\alpha + \gamma v + 1\big) - \frac{w \ln w}{\beta} - \frac{1}{\beta}\exp(\alpha + \gamma v) + v. \tag{8}$$

Observe that $wx - e^x$ reaches its maximum for x^* such that $e^{x^*} = w$, therefore $wx - e^x - w\ln w + w \leq 0$ for all x, and this implies $m(w,v) \leq v$.

Whenever $\ln w > \alpha+\gamma v$ and $v > y$, that is for v in the interval $[y, (\ln w - \alpha)/\gamma]$ expression (8) can be solved implicitly for the (unique) level of utility $v = \psi(w,y)$ that can be achieved optimally when the wage is w and the level of unearned income is y: the indirect utility function. Direct application of the implicit function theorem determines ψ_w and ψ_y, and Roy's identity yields expression (3). Whenever the level of unearned income is large relative to the wage, i.e. the interval $[y, (\ln w - \alpha)/\gamma]$ is empty, the optimal labour supply is equal to 0, the optimal expenditure on consumption is y and the utility level is measured by y as well.

Heckman's specification can be generalised further to allow for more complex responses of the Hicksian labour supply to a change of the wage. Consider $f(x), \forall x \in \mathbb{R}$, a positive, increasing function and let $F(x)$ be a primitive of $f(x)$. Denote $f^{-1}(y)$ the inverse function of $f(x)$. A more general specification for the MRS takes the form:

$$f^{-1}(\omega(h,c)) \equiv f^{-1}(\omega(h,y^*)) = \alpha + \beta h + \gamma y^*, \tag{9}$$

and implies that the compensated labour supply function takes the form:

$$h(w,v) = \frac{f^{-1}(w) - \alpha - \gamma v}{\beta}, \tag{10}$$

with positive labour supply whenever $w > f(\alpha + \gamma v)$. Hence the function $f^{-1}(x)$ describes the behaviour of the compensated labour supply function as the wage rate changes. The steps described above allow the derivation of the expenditure function, similar to (8), involving both $F(x)$ and $f(x)$:

$$m(w,v) = \frac{1}{\beta}\big(F(f^{-1}(w)) - F(\alpha + \gamma v)\big) + v - w\frac{f^{-1}(w) - \alpha - \gamma v}{\beta}, \tag{11}$$

whenever $w > f(\alpha+\gamma v)$. Finally, for a given value of y the unearned income, expression (11) can be solved for v to yield either the indirect utility function or the direct utility function.

An Alternative Specification

Following on Heckman's initial proposal, consider an alternative specification for the marginal rate of substitution in term of hours of work and total expenditure on consumption goods:

$$\ln \omega(h,c) = \kappa + \lambda h + \mu c, \tag{12}$$

which like the other popular specifications depends on three parameters only. Demanding $\lambda > 0$ and $\mu \geq 0$ is enough to insure that any indifference curve is increasing and convex for all values of h and c. Note that near $h = 0$, $c = y$, $\omega(h,c)$ behaves like the MRS specified by Heckman.

Whenever $\ln(w) \geq \kappa + \mu y$, the Marshallian labour supply and expenditure on consumption functions take the explicit forms:

$$h = \frac{\ln(w) - \kappa - \mu y}{\lambda + \mu w}, \tag{13}$$

$$c = \frac{w \ln w - \kappa w + \lambda y}{\lambda + \mu w}. \tag{14}$$

I can then derive the elasticities of the labour supply relative to the wage and to the unearned income:

$$e_{h/w} = \frac{\mu w}{\lambda + \mu w}\Big(\frac{1}{\mu w h} - 1\Big), \quad e_{h/y} = -\frac{\mu y}{\lambda + \mu w}\frac{1}{h}. \tag{15}$$

The first expression implies that the labour supply exhibits backward bending (i.e. the labour supply decreases for a large enough wage) whenever earnings are large enough, i.e. if $wh > \mu^{-1}$. Hence it is easier to describe the properties of this alternative specification of the Marshallian labour supply than it was for Heckman's.

Following the method of characteristics, along an indifference curve $c(h, v(0, y^*))$ satisfies

$$\frac{dc(h, v(0, y^*))}{dh} = \exp(\kappa + \lambda h + \mu c) \tag{16}$$

with $v(h, c) = y^*$. Total expenditure on consumption is therefore such that:

$$\exp(-\mu c(h, v(0, y^*)) = -\frac{\mu}{\lambda}\exp(\kappa)\Big(\exp(\lambda h) - 1\Big) + \exp(-\mu y^*). \tag{17}$$

This suggests that the indifference curve at a positive level y^* is only defined for h such that:

$$h \leq \frac{1}{\lambda}\ln\big(1 + \frac{\lambda}{\mu}\exp(-\kappa - \mu y^*)\big), \tag{18}$$

and the RHS of this inequality can be less than T for large enough values for y^* if $\lambda > 1/T$. This upper bound is the maximum labour supply that can be compensated for by more consumption, i.e. beyond this level on a given indifference curve no wage is large enough to compensate for the dis-utility of work. Therefore on any given indifference curve as h approaches the upper bound consumption expenditure approaches infinity.

For all positive values for c and h the direct utility function takes the explicit form:

$$v(h,c) = -\frac{1}{\mu} \ln \big(\exp(-\mu c) + \frac{\mu}{\lambda} \exp(\kappa)(\exp(\lambda h) - 1)\big). \tag{19}$$

If $\ln w \geq \kappa + \mu y$, the expressions for the labour supply and the expenditure on consumption, (13) and (14), can be substituted into the expression for the direct utility function and yield an expression for the indirect utility function $\psi(w,y)$ explicitly. If instead $\ln w < \kappa + \mu y$ then $h = 0$, $c = y$ and $\psi(w,y) = y$. Assuming $\ln w \geq \kappa + \mu y$, expressions (12) and (19) can be combined to reveal in turn the compensated expenditure on consumption, $c(w,v)$, the compensated labour supply function $h(w,v)$ and then the expenditure function $m(w,v)$. I find:

$$c(w,v) = v - \frac{1}{\mu} \ln \Big(\frac{\lambda + \mu w}{\lambda + \mu \exp(\kappa + \mu v)} \Big), \tag{20}$$

$$h(w,v) = \frac{1}{\lambda} \ln \Big(w \frac{\lambda \exp(-\kappa - \mu v) + \mu}{\lambda + \mu w} \Big), \tag{21}$$

$$\begin{aligned} m(w,v) &= c(w,v) - wh(w,v) \\ &= v - \frac{1}{\mu} \ln \Big(\frac{\lambda + \mu w}{\lambda + \mu \exp(\kappa + \mu v)} \Big) - \frac{w}{\lambda} \ln \Big(w \frac{\lambda \exp(-\kappa - \mu v) + \mu}{\lambda + \mu w} \Big), \end{aligned} \tag{22}$$

and it is then easy to check that differentiating $m(w,v)$ with respect to w yields $-h(w,v)$ (Shephard's lemma).

Other specifications of this kind are possible with more or less explicit properties depending on the exact formulation. For example assume that for $h \geq 0$ and $c \geq 0$:

$$\omega(h,c) = g(h)\exp(\mu c), \tag{23}$$

where $g(h)$ is some non negative and increasing function and where $\mu \geq 0$. Let $G(h) = \int_0^h g(\eta)d\eta$ for $0 \leq h \leq T$. This specification allows for substantial flexibility in the way the marginal rate of substitution varies with the hours of work. Over the ranges of wages consistent with a positive level of hours, i.e. such that $w \geq g(0)\exp(\mu y)$, the condition $\omega(h, wh+y) = w$ defines implicitly the Marshallian labour supply function. The direct utility function, $v(h,c)$ in this case satisfies:

$$\exp(-\mu v(h,c)) = \exp(-\mu c) + \mu G(h). \tag{24}$$

Knowing the Marshallian labour supply function determines the total expenditure on consumption, and substitution in (24) defines the indirect utility function.

Together with expression (23), (24) defines the Hicksian labour supply implicitly as the solution $h(w,v)$ of the equation:

$$\exp(-\mu v) = \frac{g(h(w,v))}{w} + \mu G(h(w,v)), \tag{25}$$

and substitution of this solution into (23) yields the compensated demand for consumption

$$c(w,v) = -\frac{1}{\mu}\ln(\frac{g(h(w,v))}{w}). \tag{26}$$

The expenditure function $m(w,u)$ is then obtained directly using the budget cosntraint.

Conclusion

In this note I show that a specification of the labour supply function based on the marginal rate of substitution yields alternative labour supply models, including Heckman's, such that the direct and indirect utility functions can be recovered easily. I show furthermore that it is possible to increase the flexibility of the model specifications while preserving some closed form analytical properties. I have not addressed here the specific estimation issues these models would raise. This is the purpose of further research.

Acknowlegments

I am grateful for the comments Katharina Jenderny and Thomas Aronsson provided which clarified the presentation and improved the paper.

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