Age Dependent Discount Rates, Time Inconsistent Behavior and Welfare Measurement*

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Abstract

This paper concerns welfare measurement in a dynamic economy where the instantaneous rate of time preference is age dependent. If agents are naive and do not recognize this age dependency, their savings decisions will be time inconsistent and the purpose of this paper is to analyze how the current value Hamiltonian, which is interpretable as a measure of the comprehensive net national product in utility terms, is related to welfare in this context. The problem is addressed within a standard Ramsey model and the main result is that if the discount rate declines (increases) over time along an optimal path where net investment is positive, then the current value Hamiltonian underestimates (overestimates) a measure of the interest on the present value of future utility.

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1 Introduction

Today there exists a literature which aims to construct a comprehensive net national product (NNP) measure which can be used to measure welfare in a dynamic economy. This literature dates back to the seminal work of Weitzman (1976) who showed that under perfect foresight, and in the absence of externalities and other distortions, NNP is a static equivalent of welfare measuring interest on the value function. Underlying this result are two fundamental requirements; the economy must be on the optimal growth path and the dynamic optimization problem must not be fundamentally time dependent. The latter requirement means that neither the objective function, nor the equation(s) for the state variable(s), contain any explicit time dependency except via the social discount factor.[1] The subsequent work has then focused on how to augment the basic result in Weitzman (1976) when the above mentioned requirements do not hold.[2] One question which has not been addressed in the earlier literature is how an age dependent discount rate affects welfare measurement in a dynamic economy and the purpose of this paper is to address this issue in the context of a standard Ramsey model with infinitely lived houseolds.

As mentioned above, most of the existing work on social accounting in dynamic economies relies on the assumption that agents have a constant rate

[1] The time dependence via the discount factor can be made redundant by solving the problem in current value terms.

[2] For example, Hartwick (1990), Dasgupta and Mäler (1991), and Mäler (1991) discuss how the welfare measure must be augmented in order to take account of changes in natural resource stocks, whereas Aronsson and Löfgren (1993, 1995) focus on the role of nonattributable technological change. These mechanisms influence the measurement of welfare because they introduce additional channels by which time explicitly enters the dynamic optimization problem, e.g. via an exogenous time path of a stock of natural resources or by exogenous technological change. More recently, the literature on social accounting has focused on welfare measurement problems associated with imperfect economies where the resource allocation is suboptimal from society's point of view (see Aronsson et al 2004 and Aronsson 2008).

of time preference. Yet, the rationale for this assumption is unclear since there is little evidence from the empirical literature to support constant exponential discounting.[3] Instead, several empirical studies have found that the rate of time preference is age dependent.[4] Already Gilman (1976), who inferred personal discount rates from the propensity of employees of four nonprofit US organizations to participate in their organization's retirement plan and Black (1984), who estimated discount rates from survey questions about alternative retirement systems for the US military, found that personal discount rates decline with age. A similar conclusion was reached by Warner and Pleeter (2001) when they estimated discount rates in a natural experiment based on the separation benefit choices made by US officers and enlisted personnel (who were offered a choice between a lump-sum separation benefit or an annuity) when the U.S. military reduced its size in the early 1990´s.

An age dependent discount rate may reflect the tendency of younger agents to be more impatient than their older counterparts. A basic motivation for this argument is that agents can only grasp time periods that they can relate to. To exemplify, consider a person who is 20 years old. Assume that this person is asked to make a trade-off between today and 20 years into the future. Since a 20-year old person´s living memory only goes back approximately 15-16 years, 20 years into the future seems (more than) a lifetime away. Therefore, the 20-year old person is likely to assign a low present value to anything that happens 20 years into the future. Consider instead

[3] See Thaler (1981), Ainsle and Haslam (1992), and Loewenstein and Prelec (1992).

[4] There are also other possible explanations for why the discount rate may be a function of time. One is hyperbolic discounting (see Laibson 1997 and Barro 1999). Another is provided by Sozou (1998) and Weitzman (1998) who showed that with a constant but uncertain discount rate, the certainty-equivalent discount rate will, over time, decline to the lowest possible rate. Finally, Chichilnisky (1996), and Li and Löfgren (2000), have shown that to avoid the current generation to have a "tyranny" over future generations, the resulting programme should involve a discount rate which is a declining function of time.

a person who is 40 years old and let him/her face the same trade-off. Since this person has a living memory of approximately 35-36 years, the 40-year old person can more easily grasp a time frame of 20 years. Therefore, the 40-year old is likely to assign a higher present value to anything that happens 20 years into the future than the 20-year old. This is equivalent with stating that a 40-year old person (on average) discounts the future less hard than a 20-year old person which, in turn, implies that the personal discount rate is a function of age. Of course, this argument does not imply that the discount rate is monotonously decreasing with age. Clearly, a 90-year old is likely to discount the future much harder than a 40-year old person.

Are age dependent discount rates empirically relevant for welfare measurement? To address this question, let us first look at some stylized facts for the United States and China where, as a result of increased life-expectancy and reduced family size, the age structure has indeed changed. In the United States, the median age was 30.2 years in 1950[5] but in the year 2000 the median age had increased to 35.3, and in the 2010 census the median age was estimated to be 37.2.[6] The effects are even more dramatic in China where in 1990, the median age was 24.8 whereas in 2010 it had increased to 34.6.[7] When the age structure within a population changes, the empirical studies presented above indicate that also the average discount rate in the economy changes. This, in turn, implies that also the social discount rate, which is an aggregation of the individual discount rates, will change.

If the discount rate itself is a function of time, this will introduce an additional channel which makes the optimization problem in a dynamic economy fundamentally time dependent. This will have implications for welfare measurement for (at least) two reasons. First, with a time dependent discount rate, the time argument enters the discount function in a different way compared with when the discount rate is fixed. As a consequence the standard

[5] Shrestha and Heisler (2009).

[6] Howden and Meyer (2011).

[7] The United Nations data base; http://data.un.org.

result in the literature mentioned above, i.e. that the Hamiltonian is equal to interest on the present value of future utility (with the discount rate as the rate of interest), needs to be modified because there is no longer a single discount rate covering the entire life-cycle. Instead, there will be one discount rate associated with each point in time and this feature will be reflected in the welfare measure. Second, and potentially more important, a time dependent discount rate may give rise to time inconsistent behavior among agents. The time consistency problem arises from the possibility that cognitive limitations might render agents incapable of forecasting future changes in their rate of time preference which will make them more concerned with their future welfare when they are older.[8] The argument[9] is that psychological development over time, which causes unforeseeable changes in the rate of time preference, may create dynamic inconsistency in the sense that agents regret decisions taken when younger.[10] In market economies, where the social discount rate is an aggregate of individual rates of time preference, it follows that if aging affects individual rates of time preference, it may also alter the social rate of time preference. Shifts in the age structure of a population may therefore imply that investment plans which are optimal from the perspective of the current generation need not be optimal from the perspective of future generations. This discrepancy needs to be

[8] See Laux (2000).

[9] See Bishai (2004).

[10] There are also other possible sources for time inconsistent behavior. From the works of Strotz (1956), Pollak (1968) and Goldman (1980), it is well known that a nonconstant rate of time preference (or more precisely, a discount rate which depends on the time-distance between today and some point in the future) may create a time consistency problem. It arises if the relative valuation of utility flows at different dates changes as the planning date evolves, which implies that precommitted choices will differ from those chosen sequentially. If agents do not fully recognize the future effect of an action taken today, their ex ante valuation of a given investment will not coincide with their ex post valuation of the same investment. Conditional on that the ex post valuation is 'correct', an ex ante welfare measure must adjust for this discrepancy in valuation.

accounted for when currently observed variables (such as the components in comprehensive NNP) are used as indicators of future welfare.

Given the discussion above, the purpose of this paper is to analyze the implications of a time dependent discount rate for welfare measurement in a dynamic economy. The study is based on the standard Ramsey model with infinitely lived households (dynastic families) and the households may either recognize (time consistent behavior), or not recognize (time inconsistent behavior), that their rate of time preference will change as the age composition within the household evolves over time. In each case (time consistent or time inconsistent behavior), the corresponding welfare measure is derived. When households display a time consistent behavior, it is shown that with a time dependent discount rate, the current value Hamiltonian will be equal to a weighted sum of the present value of the future utilities, where the time dependent discount rate is the weight attached to the utility observed at each point in time. When households instead display a time inconsistent behavior, it is shown that the current value Hamiltonian underestimates (overestimates) the weighted sum of the present value of the future utilities if the discount rate decreases (increases) over time along an optimal path where net the investment is positive. It is also shown how the current value Hamiltonian at a given point in time needs to be adjusted to fully reflect the actual future welfare when the households act in a time inconsistent way.

The outline of the paper is as follows. In section 2, the basic model is presented. Section 3 concerns welfare measurement when agents are time consistent whereas section 4 concerns welfare measurement when agents make time inconsistent savings decisions. The paper is concluded in Section 5.

2 The Basic Model

Consider a decentralized economy made up of identical firms and identical households where both the number of firms and the number of households is normalized to one. The household supplies one unit of labor inelastically on the labor market whereas the firm uses labor and capital as inputs in the production. The production function is characterized by constant returns to scale and the per capita production function (i.e. the production per household) is given by $f(k(t))$, where $k(t)$ is capital, t is time and where the notation of the fixed input of labor is suppressed. At each point in time the firm hires capital and labor in such quantities that the standard necessary conditions are satisfied

$$r(t) = f_k(k(t)) \tag{1}$$

$$w(t) = f(k(t)) - k(t) \cdot f_k(k(t)) \tag{2}$$

where $r(t)$ is the interest rate and $w(t)$ is the wage.

Turning to the household, its budget constraint is given by

$$\dot{k}(t) = r(t) \cdot k(t) + w(t) - c(t) \tag{3}$$

where $c(t)$ is consumption and $\dot{k}(t) = dk(t)/dt$. The budget constraint satisfies the No Ponzi Game (NPG) condition

$$\lim_{T \to \infty} k(T) \cdot e^{-\int_t^T r(z) \cdot dz} = 0 \tag{4}$$

and by combining equations (1) - (3) we obtain the economy-wide resource constraint

$$\dot{k}(t) = f(k(t)) - c(t) \tag{5}$$

At each point in time the household derives utility from consumption. The instantaneous utility function is denoted $u(c(t))$ and it is increasing and concave in $c(t)$. The household can be viewed as a dynastic family and as time goes by, the composition between young and old members within the household changes. This change in the age structure will cause the household´s discount rate to change which means that the discount rate will be a function of time; $\theta(t)$. If the age of the average member within the household increases over time, then (given the discussion in the introduction) the household becomes less impatient over time. In this case, the discount rate satisfies $\dot{\theta}(t) < 0$ and $\lim_{t\rightarrow\infty} \theta(t) = \bar{\theta}^{\min} > 0$, where $\dot{\theta}(t) = d\theta(t)/dt$ and where the limit ensures that the discount rate will not approach zero. If, on the other hand, the age of the average member within the household decreases over time, then the household becomes more impatient over time. In this case, the discount rate satisfies $\dot{\theta}(t) > 0$ and $\lim_{t\rightarrow\infty} \theta(t) = \bar{\theta}^{\max}$, where the limit ensures that the discount rate will not become infinitely large. Finally, since there is only one representative household in this model economy, there is no distinction between the representative household´s discount rate and the social discount rate.

To determine the time paths of consumption and capital, it is necessary to characterize the household's dynamic optimization problem. A key question in this context is if the household recognizes that the instantaneous discount rate will change over time? A household which does recognize this will be referred to as sophisticated, whereas a household which fails to take this into account will be referred to as naive. The latter type of household treats the discount rate observed at a given point in time t, $\theta(t)$, as exogenous and uses this discount rate to discount all future points in time in its present value calculations. Since the consumption/saving choices will depend on whether the household is sophisticated or naive, each case will be analyzed in turn and we begin with the sophisticated household.

3 Welfare Measurement with a Sophisticated Household

A sophisticated household recognizes that its discount rate will change over time. This means that the household´s intertemporal objective function at time t can be written as

$$U(t) = \int_t^T u(c(s)) \cdot e^{-\int_t^s \theta(z) \cdot dz} \cdot ds \qquad (6)$$

where T is an upper time limit. At time t, the current value Hamiltonian corresponding to the sophisticated household's optimization problem can be written as[11]

$$H(t) = u(c(t)) + \lambda(t) \cdot [f(k(t)) - c(t)] \tag{7}$$

where $\lambda(t)$ is the current value shadow price of capital. Conditional on the initial stock of capital, the Maximum Principle requires that the following conditions must hold at each point in time $s \in (t, T)$ along an optimal path

$$u_c(c(s)) - \lambda(s) = 0 \tag{8}$$

$$\dot{\lambda}(s) - \theta(s) \cdot \lambda(s) + \lambda(s) \cdot f_k(k(s)) = 0 \tag{9}$$

where $u_c = du/dc$, $f_k = df/dk$ and $\dot{\lambda} = d\lambda/dt$. Let the superindex "*" denote the value of an economic variable along an optimal time path towards the steady-state. In the steady-state, which the economy approaches as $T \to \infty$, the discount rate is constant. Depending on whether $\theta(t)$ is an increasing or a decreasing function of time before the steady-state is reached, the discount rate in the steady-state, $\tilde{\theta}$, where "$\sim$" is used to denote a steady-state value, will either be $\tilde{\theta} = \bar{\theta}^{\min}$ or $\tilde{\theta} = \bar{\theta}^{\max}$. The steady-state values for $\tilde{c}$, $\tilde{k}$ and $\tilde{\lambda}$ are implicitly determined by equations (5), (8) and (9) in the special case where $\theta(t) = \tilde{\theta}$ and where $\dot{\lambda}(s) = \dot{k}(t) = 0$. Before the steady-state is reached, equations (5), (8) and (9) implicitly define optimal time paths for consumption, $c^*(s)$, shadow price, $\lambda^*(s)$, and capital, $k^*(s)$. These functions explicitly take into account that the discount rate is time dependent. To illustrate the latter point, let us solve the differential equation for $\lambda^*(t)$. By using that the current value shadow price approaches $\tilde{\lambda}$ as $T \to \infty$, equation (9) implies

$$\lambda^*(t) = \tilde{\lambda} \cdot e^{\int_t^{\infty}[f_k(k(s)) - \theta(s)] \cdot ds} \tag{10}$$

From equation (10) we see that the time dependency of the discount rate is explicitly taken into account in the valuation of capital at time t.

[11] Since there are no distortions in the decentralized economy, one can directly use the capital accumulation constraint that was defined in equation (5) when solving the household´s problem.

The Hamiltonian along an optimal time path has certain envelope properties. A classical result in welfare economics, which was first demonstrated by Weitzman (1976), is that in the absence of externalities and imperfections, the Hamiltonian is a static equivalent to the present value of future utility in the sense that the Hamiltonian is equal to the interest on the present value of future utility. To see whether this result also holds in the present setting, let us differentiate equation (7) w.r.t. time and use equations (5), (8) and (9) to simplify the resulting expression. This produces

$$\frac{dH^*}{dt} = \theta(t) \cdot [H^*(t) - u(c^*(t))] \tag{11}$$

If we use use the integrating factor $e^{-\int \theta(z) \cdot dz}$ to solve this differential equation over the time interval (t, T), and if we use that

$$\lim_{T \to \infty} H^*(T) \cdot e^{-\int_t^T \theta(z) \cdot dz} = 0 \tag{12}$$

the following result is immediately available;

Proposition 1: *If the instantaneous discount rate is a continuous function of time and if the representative household correctly anticipates how the discount rate will change over time then, at an arbitrary point in time t, the relationship between the current value Hamiltonian along the equilibrium path and the present value of future utility is given by*

$$H^*(t) = \int_t^\infty \theta(s) \cdot u(c^*(s)) \cdot e^{-\int_t^s \theta(z) \cdot dz} \cdot ds \tag{13}$$

To interpret this result, observe first that if the instantaneous discount rate would be constant, then equation (13) simplifies to

$$H^*(t) = \theta \cdot \int_t^\infty u(c^*(s)) \cdot e^{-\theta \cdot (s-t)} \cdot ds \tag{14}$$

Since the current value Hamiltonian can be interpreted as a measure of the comprehensive net national product (NNP) in utility terms,[12] equation (14)

[12] See, for example, Aronsson et al (1997), Li and Löfgren (2002), and Weitzman (2001, 2003).

implies that the comprehensive NNP is equal to the interest on the present value of future utility. This corresponds to the classical result derived by Weitzman (1976). Returning to the result summarized in Proposition 1, it states that the classical result presented in equation (14) needs to be modified when the instantaneous discount rate is time dependent. From equation (13) it follows that the modification is that the level of utility at each point in time has to be weighted by the discount rate observed at that point in time. As such, the integral on the right hand side (RHS) of equation (13) can be viewed as a weighted sum of the future utilities, with the time dependent discount rate, $\theta(s)$, being the weight. Although the current value Hamiltonian in this context will not be proportional to the present value of future utility, the current value Hamiltonian can nevertheless be interpreted as a static equivalent to welfare in the sense that for a given time path of the instantaneous discount rate, an increase (decrease) in the present value of future utility will lead to an increase (decrease) in the current value Hamiltonian, albeit not proportionally.

4 Welfare Measurement with a Naive Household

A naive household does not take into account that the discount rate will change over time. This means that the discount rate observed at time t, $\theta(t)$, is expected to remain unchanged at all points in time $s \in (t, T)$. The intertemporal objective function that the naive household aims to maximize at time t can then be specified as follows

$$U(t) = \int_t^T u(c(s)) \cdot e^{-\theta(t)\cdot(s-t)} \cdot ds \tag{15}$$

The necessary conditions associated with the naive household's consumption/saving choice at time t will be given by

$$u_c(c(s)) - \lambda(s) = 0 \tag{16}$$

$$\dot{\lambda}(s) - \theta(t) \cdot \lambda(s) + \lambda(s) \cdot f_k(k(s)) = 0 \tag{17}$$

where the superindex "$\circ$" will be used to denote the value of an economic variable when the household is naive. For $s = t$, equations (16) and (17) determine the naive household´s actual consumption choice, $c^{\circ}(t, \theta(t))$, and the naive household´s actual valuation of capital, $\lambda^{\circ}(t, \theta(t))$, at time t. For all $s > t$, equations (5), (16) and (17) instead define time paths for consumption, $c^{\circ}(s, \theta(t))$, shadow price, $\lambda^{\circ}(s, \theta(t))$, and capital, $k^{\circ}(s)$, that the household at time t anticipates to observe in the future conditional on that the discount rate is fixed at the level $\theta(t)$. However, if the future discount rate, $\theta(s)$, differs from the discount rate observed at time t, then the household´s actual valuation of capital at time s will differ from the valuation that the household at time t anticipated itself to have at time s, i.e. $\lambda^{\circ}(s, \theta(s)) \neq \lambda^{\circ}(s, \theta(t))$. In this situation, the household at time s updates its consumption/saving plan and the actual consumption choice made at time s will differ from the consumption choice that was expected at time t for time s, i.e. $c^{\circ}(s, \theta(s)) \neq c^{\circ}(s, \theta(t))$. This updating procedure is repeated continuously as long as $\theta(s)$ is changing over time and for the analysis below, we note that the function $c^{\circ}(s, \theta(s))$ defines the actual consumption choice at time s while the function $\lambda^{\circ}(s, \theta(s))$ defines the actual valuation of capital at time s.

Let us now ask two questions. First, what is an appropriate measure of welfare when the household is naive and continuously updates its consumption/saving decisions? Second, is it possible to relate the naive household´s current value Hamiltonian at time t to (an appropriate measure of) welfare? As for the first question, the view in this paper is that the measure of welfare should reflect the actual preferences that the household has, and the actual choices that the household makes, at each point in time. This means that the discount rate actually observed at time s, $\theta(s)$, should be used to discount the instantaneous utility that the household actually experiences at that point in time, $u(c^{\circ}(s, \theta(s)))$. These arguments imply that the present value calculation of future utility that appears in equation (6) is an appropriate measure of welfare also when the household is naive. As such, the welfare criterion when the household is naive is the same as when the household is sophisticated. We will refer to this as the actual welfare (in contrast to the naive household´s anticipated welfare which is reflected in equation 15).

To address the second question, we begin by writing the naive household´s current value Hamiltonian at time t as

$$H^{\circ}(t) = u\left(c^{\circ}(t, \theta(t))\right) + V^{\circ}(t, \theta(t)) \tag{18}$$

where $V^{\circ}(t, \theta(t)) = \lambda^{\circ}(t, \theta(t)) \cdot \dot{k}^{\circ}(t, \theta(t))$ is the naive household´s valuation (in utility terms) of the investment made at time t. Differentiating $H^{\circ}(t)$ w.r.t. time produces

$$\frac{dH^{\circ}(t)}{dt} = \frac{\partial u^{\circ}}{\partial c^{\circ}} \cdot \frac{\partial c^{\circ}}{\partial t} + \frac{\partial u^{\circ}}{\partial c^{\circ}} \cdot \frac{\partial c^{\circ}}{\partial \theta} \cdot \frac{\partial \theta}{\partial t} + \frac{\partial V^{\circ}}{\partial t} + \frac{\partial V^{\circ}}{\partial \theta} \cdot \frac{\partial \theta}{\partial t} \tag{19}$$

Next, substitute the definition $\dot{k}^{\circ}(t, \theta(t)) = f(k^{\circ}(t)) - c^{\circ}(t, \theta(t))$ into $V^{\circ}(t, \theta(t)) = \lambda^{\circ}(t, \theta(t)) \cdot \dot{k}^{\circ}(t, \theta(t))$ and differentiate the resulting expression w.r.t. t but conditional on θ. This produces

$$\frac{\partial V^{\circ}}{\partial t} = \frac{\partial \lambda^{\circ}}{\partial t} \cdot \frac{\partial k^{\circ}}{\partial t} + \lambda^{\circ} \cdot \left(f_k^{\circ} \cdot \frac{\partial k^{\circ}}{\partial t} - \frac{\partial c^{\circ}}{\partial t} \right) \tag{20}$$

By using equations (16), (17) and (20) together with $\lambda^{\circ} \cdot \dot{k}^{\circ} = H^{\circ} - u^{\circ}$, we can rewrite equation (19) to read

$$\frac{dH^{\circ}}{dt} = \theta(t) \cdot \left[H^{\circ}(t) - u\left(c^{\circ}(t, \theta(t))\right)\right] + \gamma^{\circ}(t, \theta(t)) \cdot \frac{\partial \theta(t)}{\partial t} \tag{21}$$

where

$$\gamma^{\circ}(t, \theta(t)) = \frac{\partial u^{\circ}\left(c^{\circ}(t, \theta(t))\right)}{\partial c^{\circ}} \cdot \frac{\partial c^{\circ}(t, \theta(t))}{\partial \theta} + \frac{\partial V^{\circ}(t, \theta(t))}{\partial \theta} \tag{22}$$

Multiply equation (21) by $U = e^{-\int \theta(z) \cdot dz}$ and then integrate over the time interval (t, T). If we use[13]

$$\lim_{T\to\infty} H^{\circ}(T) \cdot e^{-\int_t^T \theta(z)\cdot dz} = 0 \tag{23}$$

to simplify the resulting expression, we obtain the following equation

$$H^{\circ}(t) = \int_t^{\infty} \theta(s) \cdot u\left(c^{\circ}(s, \theta(s))\right) \cdot e^{-\int_t^s \theta(z)\cdot dz} \cdot ds - \Psi^{\circ}(t) \tag{24}$$

where

$$\begin{aligned} \Psi^{\circ}(t) = & \int_t^{\infty} \frac{\partial u^{\circ}\left(c^{\circ}(s, \theta(s))\right)}{\partial c^{\circ}} \cdot \frac{\partial c^{\circ}(s, \theta(s))}{\partial \theta} \cdot \frac{\partial \theta(s)}{\partial s} \cdot e^{-\int_t^s \theta(z)\cdot dz} \cdot ds \\ & + \int_t^{\infty} \frac{\partial V^{\circ}(s, \theta(s))}{\partial \theta} \cdot \frac{\partial \theta(s)}{\partial s} \cdot e^{-\int_t^s \theta(z)\cdot dz} \cdot ds \end{aligned} \tag{25}$$

Equation (24) shows that the current value Hamiltonian, and hence the comprehensive NNP, observed at time t will not represent a static equivalent of the actual welfare[14] when the household is naive and fails to recognize that the discount rate will change over time. This behavioral naivety needs to be taken into account when we relate $H^{\circ}(t)$ to the actual welfare and this is accomplished by subtracting the corrective term $\Psi^{\circ}(t)$ from the RHS of equation (24). This term captures that as the discount rate (unexpectedly from the naive household´s point of view) changes over time, then the actual future time path of consumption and the actual future time path of the valuation of investment will, over time, deviate from the corresponding anticipated time paths. To see this more clearly, observe that the actual changes in $u^{\circ}\left(c^{\circ}(s, \theta(s))\right)$ and $V^{\circ}(s, \theta(s))$ that occur between two points in time s and $s + ds$ can be decomposed into the anticipated change that occurs conditional on the observed discount rate and the unanticipated change

[13]Note that we use the necessary conditions associated with the naive housheold´s **actual** consumption decision and the naive household´s **actual** valuation of capital at time s when we integrate and solve the differential equation in (21) over all $s \in (t, T)$. This follows because we use the functions $c^{\circ}(s, \theta(s))$ and $\lambda^{\circ}(s, \theta(s))$ (instead of the functions $c^{\circ}(s, \theta(t))$ and $\lambda^{\circ}(s, \theta(t))$) when we derive equation (24).

[14]Recall from the discussion following Proposition 1 that the integral on the RHS of equation (24) is a representation of the actual welfare in the sense that it is a weighted sum of the future utilites.

which arises as the discount rate changes over the time interval ds. By using the following first-order Taylor approximations, these changes are (for small ds) given by

$$du^{\circ} \approx \underbrace{\frac{\partial u^{\circ}}{\partial c^{\circ}} \cdot \frac{\partial c^{\circ}}{\partial s} \cdot ds}_{du^{\circ,a}} + \underbrace{\frac{\partial u^{\circ}}{\partial c^{\circ}} \cdot \frac{\partial c^{\circ}}{\partial \theta} \cdot \frac{\partial \theta}{\partial s} \cdot ds}_{du^{\circ,u}} \tag{26}$$

$$dV^{\circ} \approx \underbrace{\frac{\partial V^{\circ}}{\partial s} \cdot ds}_{dV^{\circ,a}} + \underbrace{\frac{\partial V^{\circ}}{\partial \theta} \cdot \frac{\partial \theta}{\partial s} \cdot ds}_{dV^{\circ,u}} \tag{27}$$

The terms $du^{\circ,a}$ and $dV^{\circ,a}$ are the anticipated changes in u° and V°, respectively, and it is not necessary to account for them in the corrective term $\Psi^{\circ}(t)$. However, this argument does not apply to the terms $du^{\circ,u}$ and $dV^{\circ,u}$ which capture the unanticipated effects on u° and V° that arise as the discount rate changes over time. These effects need to be accounted for when $H^{\circ}(t)$ is related to the actual welfare and by substituting the defintions of $du^{\circ,u}$ and $dV^{\circ,u}$ into equation (25), it follows that the corrective term $\Psi^{\circ}(t)$ is equal to the discounted sum of $du^{\circ,u}$ and $dV^{\circ,u}$ over all future points in time. As such, the corrective term $\Psi^{\circ}(t)$ is forward looking and accounts for the future welfare effects that the unanticipated future changes in the discount rate will have on u° and V°.

Can the expression for $\Psi^{\circ}(t)$ in equation (25) be simplified? The answer is yes and to do this simplification, let us first observe that from $V^{\circ}(t, \theta(t)) = \lambda^{\circ}(t, \theta(t)) \cdot \dot{k}^{\circ}(t, \theta(t))$ it follows that

$$\frac{\partial V^{\circ}}{\partial \theta} \cdot \frac{\partial \theta}{\partial s} = \frac{\partial \lambda^{\circ}}{\partial \theta} \cdot \dot{k}^{\circ} \cdot \frac{\partial \theta}{\partial s} + \lambda^{\circ} \cdot \frac{\partial \dot{k}^{\circ}}{\partial \theta} \cdot \frac{\partial \theta}{\partial s} \tag{28}$$

Substituting this expression into equation (25), while we simultaneously use that[15] $\partial \dot{k}^{\circ}/\partial \theta = -\partial c^{\circ}/\partial \theta$, produces

$$\Psi^{\circ}(t) = \int_{t}^{\infty} \left[\left(\lambda^{\circ} - \frac{\partial u^{\circ}}{\partial c^{\circ}} \right) \cdot \frac{\partial \dot{k}^{\circ}}{\partial \theta} + \frac{\partial \lambda^{\circ}}{\partial \theta} \cdot \dot{k}^{\circ} \right] \cdot \frac{\partial \theta}{\partial s} \cdot e^{-\int_{t}^{s} \theta(z) \cdot dz} \cdot ds \tag{29}$$

Since $u_{c}^{\circ}(c^{\circ}(s, \theta(s))) = \lambda^{\circ}(s, \theta(s))$ holds at each point in time s, the first term inside square brackets in equation (29) will vanish. This means that

[15] This equality follows from the definition that $\dot{k}^{\circ}(t, \theta(t)) = f(k^{\circ}(t)) - c^{\circ}(t, \theta(t))$.

at each point in time $s \in (t, \infty)$, the welfare effect of the unanticipated change in the investment $(\lambda^\circ \cdot \partial \dot{k}^\circ / \partial \theta)$ is exactly offset by the corresponding welfare effect associated with the unanticipated change in the consumption $(\partial u^\circ / \partial c^\circ \cdot \partial \dot{k}^\circ / \partial \theta)$. Using this envelope property to simplify $\Psi^\circ(t)$ in equation (24) produces the following result;

Proposition 2: *If the instantaneous discount rate is a continuous function of time, and if the representative household is naive in the sense that it believes that the discount rate observed at an aritrary point in time t will remain unchanged in the future, then the relationship between the current value Hamiltonian observed at time t and the present value of future utility is given by*

$$\begin{aligned} H^\circ(t) = & \int_t^\infty \theta(s) \cdot u(c^\circ(s, \theta(s))) \cdot e^{-\int_t^s \theta(z) \cdot dz} \cdot ds \\ & - \int_t^\infty \dot{k}^\circ(s, \theta(s)) \cdot \frac{\partial \lambda^\circ(s, \theta(s))}{\partial \theta} \cdot \frac{\partial \theta}{\partial s} \cdot e^{-\int_t^s \theta(z) \cdot dz} \cdot ds \end{aligned} \tag{30}$$

We are now able to provide a more precise interpretation of what the corrective term actually captures. It can be summarized as follows: As the discount rate changes over time (captured by $\partial \theta / \partial s$), the valuation of one unit of investment will change (captured by $\partial \lambda^\circ / \partial \theta$). The discounted sum of the changes in the valuation of one unit of investment multiplied by the size of the investment $(\dot{k}^\circ)$ made at each point in time provides us with an estimate of how much the naive household will wrongly anticipate the future welfare effects of investment.

Let us finally consider under what circumstances the corrective term is positive or negative. From the expression in the second row of equation (30), it follows that the sign of the corrective term depends on the signs of $\dot{k}^\circ$, $\dot{\theta} = \partial \theta / \partial s$ and $\partial \lambda^\circ / \partial \theta$. Since the last partial derivative is negative,[16] there are four possible scenarios which can be summarized as follows;

[16] Solving the differential equation in (17) conditional on that $\theta(t)$ is unchanged at all

Corollary: *The current value Hamiltonian underestimates the actual welfare* ($\Psi^{\circ}(t) > 0$) *if (i)* $\dot{k}^{\circ} > 0$ *<u>and</u>* $\dot{\theta} < 0$*, or if (ii)* $\dot{k}^{\circ} < 0$ *<u>and</u>* $\dot{\theta} > 0$*.*

The current value Hamiltonian overestimates the actual welfare ($\Psi^{\circ}(t) < 0$) *if (iii)* $\dot{k}^{\circ} < 0$ *<u>and</u>* $\dot{\theta} < 0$*, or if (iv)* $\dot{k}^{\circ} > 0$ *<u>and</u>* $\dot{\theta} > 0$*.*

Consider the ´standard´ case where the economy is on a positive trajectory towards steady state ($\dot{k}^{\circ} > 0$) and where the population is ageing and therefore becomes less impatient over time ($\dot{\theta} < 0$). This scenario corresponds to part (i) in the Corollary and implies that the current value Hamiltonian underestimates the actual welfare. The reason is that since the naive household does not recognize the future reduction in the discount rate, the future valuation of an investment made today will be underestimated. Along a trajectory where the net investment is positive, this cognitive limitation will cause the naive household to underestimate the future welfare effects of investment. If the discount rate instead would increase over time ($\dot{\theta} > 0$), then the future valuation of investment would be overestimated. Along a trajectory where the net investment is positive, the naive household would therefore overestimate the future welfare effects of investment. This corresponds to part (iv) in the Corollary. Parts (ii) and (iii) in the Corollary can be interpreted analogously.

5 Conclusions

This paper addresses the question of how a time dependent discount rate will affect welfare measurement in a dynamic economy. Two scenarios are considered. In the first, the private agents are sophisticated and recognize how the discount will change in the future. In the second scenario, the

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future points in time, and conditional on that a steady-state is reached, produces

$$\lambda^{\circ}(t, \theta(t)) = \tilde{\lambda} \cdot e^{\int_t^T [f_k(k(s)) - \theta(t)] \cdot ds}$$

from which it follows that $\partial \lambda^{\circ} / \partial \theta(t) < 0$.

agents are naive in the sense that they do not recognize that the discount rate will change over time and this naivety will give rise to time inconsistent consumption/saving plans. The problem is addressed within a standard Ramsey model and two broad conclusions and be drawn from the analysis:

(i) If the agents are sophisticated then the comprehensive NNP can still be interpreted as an indicator of welfare. However, the classical result derived by Weitzman (1976), where the comprehensive NNP is equal to the interest on the present value of future utility, is slightly modified. When the discount rate is time dependent, the utility observed at each point in time needs to be weighted by the corresponding time dependent discount rate in the present value calculation of future utility.

(ii) If the agents do not recognize that the discount rate is time dependent, then the comprehensive NNP will not (in general) represent a static equivalent to the actual welfare. If the economy is on a growth trajectory (i.e. the per capita capital stock increases over time) and if the discount rate is declining (increasing) over time because the population is ageing (becoming younger) in the sense that the median household member becomes older (younger), then the comprehensive NNP observed at a given point in time will underestimate (overestimate) the actual future welfare. On the other hand, if the economy experiences negative growth in the sense that the per capita capital stock decreases over time and the discount rate is declining (increasing) over time, then the comprehensive NNP observed at a given point in time will overestimate (underestimate) the actual future welfare.

A natural question is whether it is possible to generalize the results derived in this paper into a setting where agents differ w.r.t. the degree of naivety (i.e. some agents are sophisticated whereas other are naive)? I leave this and other extensions for future research.

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