Welfare Theory: History and Modern Results*

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November 2007

1. Introduction

It is reasonable to say that Adam Smith (1776) has played an important role in the development of welfare theory. The reasons are at least two. In the first place, he created the invisible hand idea that is one of the most fundamental equilibrating relations in Economic Theory; the equalization of rates of returns as enforced by a tendency of factors to move from low to high returns through the allocation of capital to individual industries by self-interested investors. The self-interest will result in an optimal allocation of capital for society. He writes: "Every individual is continually exerting himself to find out the most advantageous employment for whatever capital he can command. It is his own advantage, indeed, and not that of society, which he has in view. But the study of his own advantage naturally, or rather necessarily leads him to prefer that employment which is most advantageous to society".

He does not stop there but notes that what is true for investment is true for economic activity in general. "Every individual necessarily labours to render the annual revenue of the society as great as he can. He generally, indeed, neither intends to promote the public interest, nor knows how much he is promoting it". He concludes: "It is not from the benevolence of the butcher, the brewer, or the baker, that we expect our dinner, but from the regard of their own interest". The most famous line is probably the following: The individual is "led by an invisible hand to promote an end which was no part of his intention". The invisible hand is competition, and this idea was present already in the work of the brilliant and undervalued Irish economist Richard Cantillon. He sees the invisible hand as embodied in a central planner, guiding the economy to a social optimum[1].

The second reason why Adam Smith played an important role in the development of welfare theory is that, in an attempt to explain the "Water and Diamond Paradox", he came across an important distinction in value theory. At the end of the fourth chapter of the first book in Adam Smith's celebrated volume *The Wealth of Nations (1776),* he brings up a valuation problem that is usually referred to as *The Value Paradox*[2]. He writes

"The word VALUE, it is to be observed, has two different meanings, and sometimes expresses the utility of some particular object, and sometimes the power of purchasing other goods which the possession of that object conveys. The one may be called "value in use"; the other, "value in exchange". The things which have the greatest value in use have frequently little or no value in exchange; and, on the contrary, those which have the greatest value in exchange have frequently little or no value in use. Nothing is more useful than water: but it will purchase scarce anything; scarce anything can be had in exchange for it. A diamond on the contrary, has scarce any value in use; but a very great quantity of other goods may frequently be had in exchange for it."[3]

He is unable to credibly resolve the paradox - although he uses three chapters to convince the reader that it can be resolved by the components of the natural price, i.e., essentially the notion that the long-run price is determined by the production costs. Some of the reasons behind the "failure" are not farfetched. Adam Smith was aware of supply and demand without being able to produce anything fresh about the fundamental ideas upon which these concepts rest. He was not aware of the idea to model the total utility value of consumption in terms of a utility function, and the related idea of assuming that the utility function exhibits a declining marginal utility.

Rather, it was Jules Dupuit (1844) and Heinrich Gossen (1854), who founded the modern utilitarian framework in Economics. Adam Smith's distinction between value-in-use and value-in-exchange nevertheless contain a non-trivial insight, which is fundamental for the answer to the Water and Diamond Paradox. The value in exchange is not enough to measure welfare. As Dupuit pointed out, you also need the consumer surplus which is the difference between the total surplus and the value in exchange.

The next important step in the development of welfare theory was unmistakably achieved by Leon Walras (1874). He introduced the full fledged general equilibrium system based on the fundamental principles of utility maximization and profit maximization (firms). He showed that only relative prices could be determined, since only relative prices affects the actions of firms and consumers, which means that an *n*-goods system has only *n-1* independent equations. A point that Walras expressed by picking one good as numeraire which conveniently can be given unit price. He also showed that the budget constraint of each consumer and the objective functions facing the firms together imply that the market value of supply equals the market value of demand independent of the price vector. Most modern welfare results are in one way or another connected to the competitive general equilibrium system.

However, what was still missing after Walras had done the unifying job in his magnum opus *Element d´Economie Politique Pure*, was an idea how to rank different general equilibrium allocations. Economists had long been aware of that distributional issues matter, and that Jeremy Bentham’s idea of maximizing utility of the maximum number of people typically involves one maximum too many to be feasible. It was Vilfredo Pareto, who took the distributional issue quite a bit further. He made two key contributions to existing theory. First, he realized that it was not necessary, as was implicit in 19:th century Economics, that utility was cardinal, i.e., measurable in the manner which makes interpersonal utility comparisons possible. It was enough for deriving demand functions that utility was ordinal, i.e., only the individuals’ rankings of different commodity bundles matter (utilities are ranked not their differences). As Pareto (1909) expressed it in the second edition of his Manuel d’Economie Politique[4]: “The individual may disappear, provided he leaves us that photograph of his tastes”. More formally, any monotone transformation of the utility function will result in the same demand vector[5]. The second and most important contribution is a partial ordering that admitted inter-personal welfare comparisons. He proposed that welfare increases if some people gain and nobody loses. Welfare declines if some people lose and nobody gains. If some gain and some lose, the welfare change is ambiguous, no verdict. This partial ordering was later called the Pareto criterion.

Clearly, it is always favorable to exhaust all mutually advantageous trades, and the resulting state is called Pareto optimal. Pareto realized that there are typically many such states starting from a given allocation of the initial resources. To illustrate Pareto optimality he could have used the concept of the contract curve that was invented 25 years earlier by Francis Ysidro Edgeworth (1881), but he used both the contract curve and a box that strangely enough today is called an Edgeworth box, perhaps since it encompasses the contract curve.

Now the time had come to produce modern welfare theory, and the man who did it was a student of Alfred Marshall and a classmate of John Maynard Keynes at Cambridge, England, namely Arthur Cecil Pigou. In Wealth and Welfare (1912) he discussed how a judicious government can increase welfare. The full fledged version of the modern welfare theory was fleshed out in The Economics of Welfare (1920). Apart from containing most of the relevant welfare results that follow from the Pareto criterion and Walras' general equilibrium system it also, by introducing externalities and showing how they can be handled by environmental taxes, foreshadowing modern environmental economics by almost 50 years. The welfare optimizing (improving) taxes under externalities are today called Pigouvian taxes[6]. The driving force behind Pigou's contribution to welfare economics is his distinction between private and social cost. If they coincide the invisible hand, driven by self-interest, will tend to bring about an efficient allocation of resources (first best). In reality, with existing externalities (both positive and negative), there is room for improvement of the allocation by e.g. environmental taxes or subsidies. Another of Pigou's contributions, A Study in Public Finance (1947), contains fundamental insights with respect to public good provision; in particular, how the use of distortionary taxation modifies the cost benefit analysis underlying the supply of public goods. These ideas were later developed by other researchers into the concept of 'marginal cost of public funds'.

Pigou's contributions were mainly dressed in prose. He was, unlike his teacher Marshall, not well educated in mathematics. He went to Cambridge to study history and literature. A full fledged stringent version of modern welfare theory had to wait until the publication of Abba P. Lerner's (1934) paper and the book The Control of Economic Resources (1944). Lerner was the first to describe the system as a whole and to show that a competitive market economy generates a Pareto optimal allocation of resources; a result known as the First Fundamental Theorem of Welfare Economics. Starting from a competitive equilibrium he shows that the conditions for an optimal allocation of consumption goods are fulfilled, as well as the condition for efficiency in production. Finally, he shows that in equilibrium there is equality between the marginal rate of substitution in consumption and marginal rate of transformation in production for each pair of goods. The reason is that both producers and consumers optimize their action facing the same prize vector. In other words, a competitive economy generates a Pareto optimal allocation of resources where both consumption and production are efficient; a formal proof of Adam Smith invisible hand conjecture. A similar proof can be found in Oskar Lange (1942), while Kenneth Arrow (1951a) uses topological methods and separating hyperplane theorems.

Lange and Taylor (1938) and Lerner (1944) also discussed the reverse result, that all Pareto optima can be supported by a price system after lump sum transfers of the initial wealth endowment. They did not produce a formal proof, but the conjecture was important for the discussion whether planned economies could reach a Pareto optimum. The first formal proof of this conjecture is probably due to Arrow (1951a), and the result is known as the Second Fundamental Theorem of Welfare Economics. We will return to these theorems below.

The Pareto criterion leaves the distributional problem unsolved. Arbraham Bergson suggested, in a paper published in 1938, that this problem can be addressed by a welfare function, which is an increasing function of the consumer's utility functions. Technically, we can now solve the resource allocation problem by maximizing the social welfare function subject to the technological constraints. The resulting allocation will be Pareto optimal, and the income distribution will be the appropriate one. However, the more specific preferences we build into the welfare function, the more relevant it will be to ask the question: Why this particular form of welfare function? Where does it come from, and does it reflect the preferences of the population in a reasonable way? This problem was approached by Arrow, who in a famous monograph first published in 1951 showed that if one starts from reasonable axioms on individual preferences and tries to aggregate them into a social ordering that fulfils similar axioms, this is impossible. At least one of the social choice axioms is violated. Most proofs (including Arrow's own) show this by proving that the non-dictatorship condition is violated. Many researchers have tried to modify the axioms to resolve the conflict between individual and social orderings, but no fully satisfactory solution has been found. The result is called Arrows Impossibility Theorem or, for that matter, the Third Fundamental Theorem of Welfare Economics.

The rest of the chapter is organized as follows. Section 2 sets out a simple static Walrasian general equilibrium model, which serves as a benchmark to be used in later sections. The principles of cost benefit analysis are dealt with in Sections 3. Section 4 addresses three fundamental issues; namely, The First and Second Welfare Theorems mentioned above, as well as introduces The Core of the Market Economy. The welfare gains from free trade are briefly discussed in Section 5, while Arrow's Impossibility Theorem is discussed in Section 6. Section 7 deals with externalities and introduces the concept of Pigouvian taxes, whereas public goods are dealt with in Section 8. Section 9 briefly discusses the area of Mechanism Design. Finally, Section 10 extends the benchmark model to a dynamic framework. This is interesting for at least two reasons. First, the extension allows us to address the time-dimension and, therefore, introduce dynamic analogues to some of the concepts addressed in earlier sections. Second, and more importantly, it enables to connect our survey on Welfare Theory to the growing literature on welfare measurement in dynamic economies.

2. A Simple Walrasian General Equilibrium model

To illustrate the efficiency properties of a Walrasian equilibrium, we will discuss the most simple Walrasian equilibrium model that involves production. We later modify the model to deal with distributional issues and the First- and Second Welfare Theorem, externalities, taxes and second best considerations. We also discuss Arrows Impossibility Theorem (The Third Welfare Theorem).

Let us start with the consumer, who has a strictly concave and twice continuously differentiable utility function denoted by

$$u = u(x, l^s) \tag{1}$$

where x is the demand for a consumption good, and l^s is the supply of labor. The utility function is increasing in consumption and decreasing in labor. The budget constraint of the consumer is

$$\pi + wl^s - px = 0 = B(x, l^s; p, w, \pi) \tag{2}$$

where *p* and *w* are the prices of consumer goods and labor. Here π is the profit income from the production sector/the firm. It enters the budget constraint, since the representative individual is assumed to own the firm. The firm has the technology/ production function

$$x^s = f(l) \tag{3}$$

where x^s is the supply of goods and *l* is the demand for labor/the input of labor. The production function is increasing in l, strictly concave and twice continuously differentiable.

The firm maximizes profit, while treating *p* and *w* as exogenous, which generates the optimal profit function

$$\pi[p,w)] = \max_{l}\{pf(l) - wl\} = px^s(p,w) - wl(p,w) \tag{4}$$

The consumer maximizes utility subject to the budget constraint, treating *p*, *w* and π as exogenous, which gives the optimal value (indirect utility) function

$$u[x(p,w,\pi), l^s(p,w,\pi)] = \max_{x,l^s}\{u(x,l^s) \,\big|\, x, l^s \in B(x,l^s; p,w,\pi)\} \tag{5}$$

Now, let us substitute the profit function into the budget constraint as well as into the demand and supply function of the consumer to obtain

$$B(p,w) = p[x(p,w) - x^s(p,w)] + w[l(p,w) - l^s(p,w)] = 0 \tag{6}$$

The first term is the value of excess demand in the market for goods, while the second term is the value of excess demand in the labor market. This equation means that *the values of excess demands sum to zero, independently of prices.* A little thought reveals that this condition holds for any number of markets and any number of consumers or firms. One can view equation (6) and its generalization as the economy's aggregate budget constraint, and the summation result corresponds to Walras' law. General equilibrium is typically defined as a situation, in which demand equals supply in all markets. Another, more stringent definition allows for excess supply in a number of markets, but in this case the equilibrium price in these markets must be zero. Equation (6) shows that if one market is in equilibrium, then the other market must also be in equilibrium. This means, in the n-market case, that there are only *n-1* independent markets but *n* prices. This is why Walras used one good as numraire with a price equal to one. The trick works since the demand and supply functions are homogenous of degree zero, i.e., doubling all prices does not change firms' and consumers' optimal decisions. Firms an consumers lack money illusion.

Now, if we scale everything with the inverse of the price for goods, we can determine the relative price, $\omega = w / p$, either by equating the demand and supply of goods or by equating the demand and supply of labor. Say that we solve for the equilibrium price by equating demand and supply in the market for goods:

$$x(\omega^*) = x^s(\omega^*) \tag{7}$$

where $\omega^* = w / p$ is the equilibrium real wage, which can be used to solve for all variables in general equilibrium.

In the diagram below we illustrate the equilibrium in our Robison Cruse Economy. On the vertical axis we measure the demand and supply of goods and on the horizontal axis the supply and demand of labor. The concave production function is illustrated by the curve OP, and the convex indifference curves are denoted A, B, and C. The arrow in the diagram points out the direction of increasing utility. The curve denoted B has a common point with the production function, and through that point the straight line $y = \pi(\omega) + \omega l$ is tangent to both the indifference curve and the production function. This straight line corresponds to the economy's aggregate budget constraint. The slope of the line is the real wage rate in equilibrium.

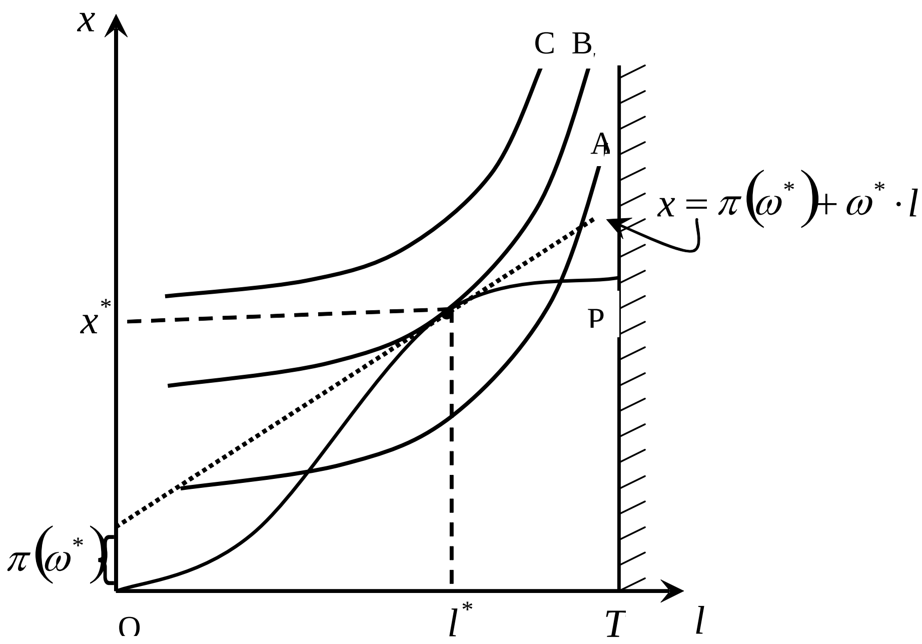

Figure 1: The Competitive Equilibrium

In Figure 1, the competitive equilibrium is given by the point (l^*, x^*), which is assumed to be unique. However, uniqueness is not generally true, and an equilibrium may not even exist (see the survey on General Equilibrium Theory for details). For the equilibrium to exist in the case under consideration it is not necessary that the utility function is strictly concave. It is enough that the set above an indifference curve is a strictly convex set, which is the case in the figure.

It is also obvious from the geometry that the equilibrium point maximizes utility. We will, however, shortly move to the First and Second Welfare Theorems. To produce more general theorems, we need a more general model than a Robinson Cruse Economy.

The Determination of the absolute price level

The reader may wonder how absolute prices are determined. Monetary theory is a well-developed branch of economics, and it would take us too far to go into any details. We briefly present a classical way to determine the price level called the Quantity Theory of Money. The underlying idea is that the quantity of money matters for normal but not for real entities. The theory is extremely old. The economic writer that is considered to be the modern "Father of the Quantity Theory" is the Italian Benardo Davanzati (1588).

In the context of our simple general equilibrium model, the following equation system generates the nominal price- and wage levels:

$$wl^* + px^* = MV$$
$$\frac{w}{p} = \omega^*$$

The left hand side of the first equation is the total value of goods in circulation, i.e., wage income plus the nominal value of consumption. The right hand side contains the stock of money times the velocity of circulation. Both entities are exogenously determined. The second equation is the definition of real wage in general equilibrium. Entities which have a star for the index are equilibrium values that are determined in the real part of the economy. It is easy to show that

$$p = \frac{MV}{\omega^* l^* + x^*}$$
$$w = p\omega^*$$

Note that the prices are proportional to the quantity of money, and that increased money supply does not affect the real entities.

3. Cost Benefit Analysis of Small Projects in General Equilibrium

The model discussed so far may seem simple, but it can be used to illustrate many fundamental insights in Welfare Economics. One key insight is determining what a cost-benefit rule will look like in a situation, in which all general equilibrium effects have been accounted for. The partial equilibrium version was first introduced by Dupuit (1844), in which he studied the value of a large project. Being an engineer educated at Ecole Polytechnique in Paris, he was confronted every day with the question of how public investments such as highways, bridges and canals should be evaluated. He looked for a measure of the utility of public works and ended up with an aggregate demand curve in a utility metrics, the area under which constituted the total willingness to pay. Taking away the value of exchange (px) left the consumer surplus. In addition, Dupuit understood the excess burden of taxation, i.e., that the tax payer is willing to pay more than the tax revenue to get rid of the tax.

The cost benefit rule that we will handle in this section is a marginal project represented by a parameter, α, which enters the model in the production function in the following manner

$$x^s = f(l;\alpha_0), \text{ with } \frac{\partial f(l;\alpha_0)}{\partial \alpha} > 0,$$

i.e., a marginal increase in the parameter from its initial value increases the productivity in the economy. There is also a cost involved, which is measured at the margin by the cost function $\frac{\partial c(\alpha_0)}{\partial \alpha} > 0$, with $c(\alpha_0) = 0$. The profit of the firm can now be written as

$$\pi = pf(l;\alpha_0) - wl - c(\alpha_0) \tag{8}$$

If we are starting at the Walrasian equilibrium with $l = l^*$ and $\alpha = \alpha_0$, the impact effect for the firm of a small increase in α can be written as

$$\frac{\partial \pi(l^*;\alpha_0)}{\partial \alpha} = p\frac{\partial f(l^*;\alpha_0)}{\partial \alpha} - \frac{\partial c(\alpha_0)}{\partial \alpha} \tag{9}$$

The reader should note that $l^* = l(p,w;\alpha_0)$ is also a function of the parameter α; however, the resulting welfare effect vanishes because l is optimally chosen (the first order condition reads $p[\partial f(l;\alpha_0)/\partial l] - w = 0$).

The optimal value function of the consumer (and technically also the whole economy since there is only one individual) can now be written

$$V = V[p(\alpha_0), w(\alpha_0), \pi(p(\alpha_0), w(\alpha_0), \alpha_0)] \tag{10}$$

The project that is inherent in equation (9) can now be evaluated by totally differentiating the value function with respect to the parameter α. We obtain:

$$\frac{dV}{d\alpha} = [\frac{\partial V}{\partial p} + \frac{\partial V}{\partial \pi}\frac{\partial \pi}{\partial p}]\frac{\partial p}{\partial \alpha} + [\frac{\partial V}{\partial w} + \frac{\partial V}{\partial \pi}\frac{\partial \pi}{\partial w}]\frac{\partial w}{\partial \alpha} + \frac{\partial V}{\partial \pi}\frac{\partial \pi}{\partial \alpha} \tag{11}$$

This expression is at first sight a bit complicated, but by using two of the most well-known results in microeconomic theory, Hotelling's Lemma and Roy's Identity, it can be greatly simplified. Hotelling's lemma tells us that the derivative of the optimal profit function with respect to the output and input prices results in the following expression:

$$\begin{aligned} \frac{\partial \pi}{\partial p} &= x^s(p,w) \\ \frac{\partial \pi}{\partial w} &= -l(p,w) \end{aligned} \tag{12}$$

In other words, differentiating with respect to the output price produces the supply function, and differentiating with respect to the input price results in the negative of the demand function for the input.

Roy's identity tells us that

$$\begin{aligned} \frac{\partial V}{\partial p} &= -\frac{\partial V}{\partial \pi} x(p,w,\pi) \\ \frac{\partial V}{\partial w} &= \frac{\partial V}{\partial \pi} l^s(p,w,\pi) \end{aligned} \tag{13}$$

where $\frac{\partial V}{\partial \pi} = \lambda$ is the marginal utility of (profit) income. It is also the Lagrange multiplier of the budget constraint in the consumer's optimization problem. In other words, the derivative of the optimal value function with respect to p equals minus the marginal utility of income times the demand function for consumption goods. The derivative of the optimal value function with respect to the wage rate equals the marginal utility of income times the labor supply function.

We can now use these insights to rewrite equation (11) in the following manner:

$$\frac{dV}{d\alpha} = \lambda(x^s - x)\frac{\partial p}{\partial \alpha} + \lambda(l^s - l)\frac{\partial w}{\partial p} + \lambda\frac{\partial \pi}{\partial \alpha} = \lambda\frac{\partial \pi}{\partial \alpha} \tag{14}$$

The last equality follows since supply equals demand in all market in general equilibrium. The result means that for a small project, the impact effect on profits given by equation (9) is enough to value a project, and all effects of the project are valued at ruling general equilibrium prices. The project is profitable (welfare improving) iff equation (14) is positive.

Hotelling's Lemma and Roy's Identity are special cases of a more general theorem called the Envelope Theorem[7]. It tells us that for any optimal value function, the derivative of the optimal value function with respect to a parameter of the problem is equal to the partial derivative of the parameter at the optimal control vector. Say that we have an optimal value function $y(\alpha) = f(x(\alpha), \alpha)$. The total derivative of this value function with respect to the parameter is obtained by

$$\frac{df}{d\alpha} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \alpha} + \frac{\partial f}{\partial \alpha} = \left.\frac{\partial f}{\partial \alpha}\right|_{x=x(\alpha)} \tag{15}$$

since $\left.\frac{\partial f}{\partial x}\right|_{x=x(\alpha)} = 0$ at the optimum. Hence, we could have moved directly to (14) from (11), but the detour was hopefully instructive. The fact that we start from a general equilibrium means that we can skip the effects that are created from changing prices. Note also that the marginal utility of income in equation (14) is positive so we can express the rule in a money metrics by dividing through with λ.

4. The First and Second Welfare Theorem

To be able to thoroughly address the two welfare theorems, we need a more general model. We need, in particular, a model which contains more than one individual. However, to avoid some technicalities, we skip production and do the analysis by equipping the agents with an initial endowment of goods that is traded. We also need more agents to make it reasonable to assume that each agent takes the prices as given; i.e., takes the market prices as independent of his/her own actions. One way to do this rigorously is to assume that there is a continuum of agents, but a short cut is to assume that they treat prices as exogenous. There are today many proofs and versions of the two welfare theorems. The analysis below follows Varian (1994).

Notation

An individual (agent) is described by his ordinal preference or his utility function $u_i(\mathbf{x_i})$. The vector $\mathbf{x_i} = (x_i^1, x_i^2, ..., x_i^n)$ is the commodity bundle that is consumed, or potentially consumed, by individual i, and x_i^j is the consumption of commodity *j*. Because individual preferences are ordinal, commodity bundles can be ranked. However, the utility metrics neither means that the utility levels tell us something about the absolute difference in utilils, nor that any two person's utility levels can be compared.

For any two commodity bundles $\mathbf{x_i}$ and $\mathbf{x_i'}$, the statement that $\mathbf{x_i}$ is preferred to or indifferent to $\mathbf{x_i'}$ is written $\mathbf{x_i} R_i \mathbf{x_i'}$. More generally (and if we neglect subscripts to simplify notations), the fact that any two alternatives can be compared can be written:

Axiom 1: For all $\mathbf{x}$ and $\mathbf{y}$, either $\mathbf{x}R\mathbf{y}$ or $\mathbf{y}R\mathbf{x}$. (the completeness axiom)

The mathematical term for a preference ordering that satisfies Axiom 1 is that it is complete (in one piece, without holes). We also need some consistence between different pairs of alternatives. More specifically if **x** is preferred or indifferent to **y** and **y** is indifferent or preferred to **z,** then **x** is indifferent or preferred to **z**. More formally:

Axiom 2: For all **x, y**, and **z, x***R***y** and **y***R***z,** imply **x***R***z.** (the transitivity axiom)

A preference ordering satisfying Axiom 2 is said to be transitive. A preference ordering satisfying both axioms is called a weak ordering. It is typically not possible to use a weak ordering to completely characterize a preference pattern. We need to introduce the possibility to handle that **x** is strictly preferred to **y**. To accomplish this, we introduce the following definition:

Definition 1: **x***P***y** means that not **y***R***x.**

$\mathbf{x}P\mathbf{y}$ is read **x** is preferred to **y.** We also introduce the following definition of indifference:

Definition 2: $\mathbf{x}I\mathbf{y}$ means $\mathbf{x}R\mathbf{y}$ and $\mathbf{y}R\mathbf{x}$ **.**

$\mathbf{x}I\mathbf{y}$ is read **x** is indifferent to **y.**

We can use the two axioms and the two definitions to prove important statements, some seemingly intuitive, about relationships within the preference ordering. For example, that $\mathbf{x}P\mathbf{y}$ and $\mathbf{y}R\mathbf{z}$, implies $\mathbf{x}P\mathbf{z}$. To prove this, suppose that $\mathbf{z}R\mathbf{x}$. From Axiom 2 it follows that $\mathbf{y}R\mathbf{z}$ and $\mathbf{z}R\mathbf{x}$, hence $\mathbf{y}R\mathbf{x}$ from transivity. However, according to Definition 1, $\mathbf{x}P\mathbf{y}$ implies not $\mathbf{y}R\mathbf{x}$, a contradiction that proves the claim.

We will for the moment use a continuous utility function to represent the ordinal preference ordering of the individuals. This means, however, that the value of utility as such has no meaning in the sense that the ranking of commodity bundles will be the same under any strictly monotone transformation of the utility function $u_i(\mathbf{x}_i)$, i.e., the ranking is preserved if we introduce a utility function $F(u_i(\mathbf{x}))$, $F^{'}(u) > 0$

Assuming n goods and N individuals with utility function $u_i(\mathbf{x}_i), i = i,..., N$, and initial endowments $\mathbf{z}_i = (z_i^1,...,z_i^n)$, we can find out how the individual chooses the most preferred commodity bundle by solving the following maximization problem:

$$\begin{aligned} &\max_{\mathbf{x_i}} u_i(\mathbf{x}_i) \\ &\textit{subject to} \\ &\mathbf{p}\mathbf{x}_i^T = \mathbf{p}\mathbf{z}_i^T \end{aligned} \tag{16}$$

Here top index T means transpose, and $\mathbf{p} = (p_1,..., p_n)$ is the ruling price vector for the *n*-goods. Since it is fairly self evident that vector products contain a transpose, we will save notational clutter by skipping this notational precision.

If the problem is well behaved, the solution will be the continuous demand functions $\mathbf{x}_i(\mathbf{p}, \mathbf{p}\mathbf{z}_i)$, which are similar to the ones introduced in section 2 above. The only thing that is different is that they contain the value of the initial endowment instead of the profit as an income argument. To be able to handle the technicalities, we need a few additional definitions.

Definition 3: An allocation $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_N)$ is a vector of consumption bundles, whose elements are the consumption vectors at the individual level.

Definition 4: A feasible allocation in the exchange economy is an allocation that is physically possible, i.e., allocations for which it holds that $\sum_{i=1}^{N} \mathbf{x}_i \leq \sum_{i}^{N} \mathbf{z}_i$.

For the particular exchange economy represented by the N maximization problems in equation (16) we define a Walrasian equilibrium in the following general manner:

Definition 5: A Walrasian equilibrium is a pair $(\mathbf{p}^*, \mathbf{x}^*\ )$ such that $\sum_{i=1}^{N} \mathbf{x}_i(\mathbf{p}^*, \mathbf{p}^*\mathbf{z}_i) \leq \sum_{i=1}^{N} \mathbf{z}_\mathbf{i}$, where $\mathbf{x}^*$ is the utility maximizing allocation under prices $\mathbf{p}^*$.

The weak inequality means that a Walrasian equilibrium is characterized by a price vector, $\mathbf{p}^*$, such that there is no market where there is an excess demand. It is, of course natural to think about a Walrasian equilibrium as a price vector that clears all markets, but Definition 5 also allows for a case where one good is undesirable. For such a good supply is strictly greater than demand.

We have previously introduced Walras' law, and that it is valid also for he exchange economy under consideration follows directly from the optimization problem in equation (16). To see this, define the excess demand vector as $\mathbf{y}(\mathbf{p}) = \sum_{i=1}^{N} [\mathbf{x}_\mathbf{i}(\mathbf{p}, \mathbf{p}\mathbf{z}_i) - \mathbf{z}_i]$. Clearly $\mathbf{p}\mathbf{y}(\mathbf{p}) = 0$, since $\mathbf{p}\mathbf{x}_i = \mathbf{p}\mathbf{z}_i$ holds for all *N* optimization problems in equation (16). Moreover, if $y_j(\mathbf{p}^*) < 0$ for some market *j*, it must hold that $\mathbf{p}_j^* = 0$. This follows directly from the definition of Walrasian equilibrium and Walras law. We can arrange for excess demand equal to zero in all markets in a Walrasian equilibrium by assuming that all goods are desirable in the sense that if $p_j = 0$

then $y_j(\mathbf{p}) > 0$. The reason is that if $y_j(\mathbf{p}^*) < 0$, then by the free good assumption $p_i^* = 0$. However, this contradicts that all goods are desirable, $y_j(p^*) > 0$.

To simplify the proofs below we will assume that all goods are desirable and define a Walrasian equilibrium $(\mathbf{x}^*, \mathbf{p}^*)$ in the following manner:

Definition 6: A Walrasian Equilibrium $(\mathbf{x}^*, \mathbf{p}^*)$ fulfills

$$(i)\ \sum_{i=1}^{N} \mathbf{x}_i = \sum_{i=1}^{N} \mathbf{z}_i$$

$$(ii)\ \textit{If } \mathbf{x}_i' \textit{ is preferred to } \mathbf{x}_i\textit{, then } \mathbf{p}\mathbf{x}_i' > \mathbf{p}\mathbf{z}_i$$

The second point means that it costs more for consumer *i* to buy a strictly preferred vector.

We will not spend time on proving the existence of equilibrium, but just mention that this can be proved by assuming that $\mathbf{y(p)}$ is a continuous function, and that Walras' law holds. We will instead introduce a definition of Pareto Efficiency.

Definition 7: A feasible allocation $\mathbf{x}$ is Pareto efficient if there is no allocation $\mathbf{x}'$ such that all agents weakly prefer $\mathbf{x}'$ to $\mathbf{x}$ and at least one agent strictly prefers $\mathbf{x}'$ to $\mathbf{x}$.

By "weakly prefer", we mean in the notation we just introduced that $\mathbf{x}' R_i \mathbf{x}$ all *i*. And that an agent j strictly prefers $\mathbf{x}'$ to $\mathbf{x}$ can be written $\mathbf{x}' P_j \mathbf{x}$.

We are now ready to produce proofs of the First and Second Fundamental Welfare Theorem.

Theorem 1: If $(\mathbf{x}, \mathbf{p})$ is a Walrasian Equilibrium, then the allocation $\mathbf{x}$ is Pareto efficient

Proof: Suppose not, and let $\mathbf{x}'$ be a feasible allocation that all agents are indifferent to or prefer to $\mathbf{x}$, and at least one agent prefer to $\mathbf{x}$. Then by property *(ii)* of the definition of Walrasian equilibrium, we have

$$\text{(A)}\ \mathbf{p}\mathbf{x}_i' \geq \mathbf{p}\mathbf{z}_i \text{ for all } i\text{, and } p\mathbf{x}_i' > p\mathbf{z}_i \text{ for at least one } i.$$

From feasibility it must hold that $\sum_i \mathbf{x}_i' = \sum_i \mathbf{z}_i$ or

(B) $p\sum_i \mathbf{x}_\mathbf{i}' = p\sum_i \mathbf{z}_\mathbf{i}$

(A)+ (B) imply

$$\mathbf{p}\sum_i \mathbf{z}_i = \mathbf{p}\sum_i \mathbf{x}_\mathbf{i}' > \mathbf{p}\sum_i \mathbf{z}_\mathbf{i}$$

which contradicts feasibility. This proves the theorem.

In Figure 2 below, we illustrate the set of Pareto optimal allocations in an Edgeworth box, where the sides denote the total initial endowments in the two goods (x, y) economy.

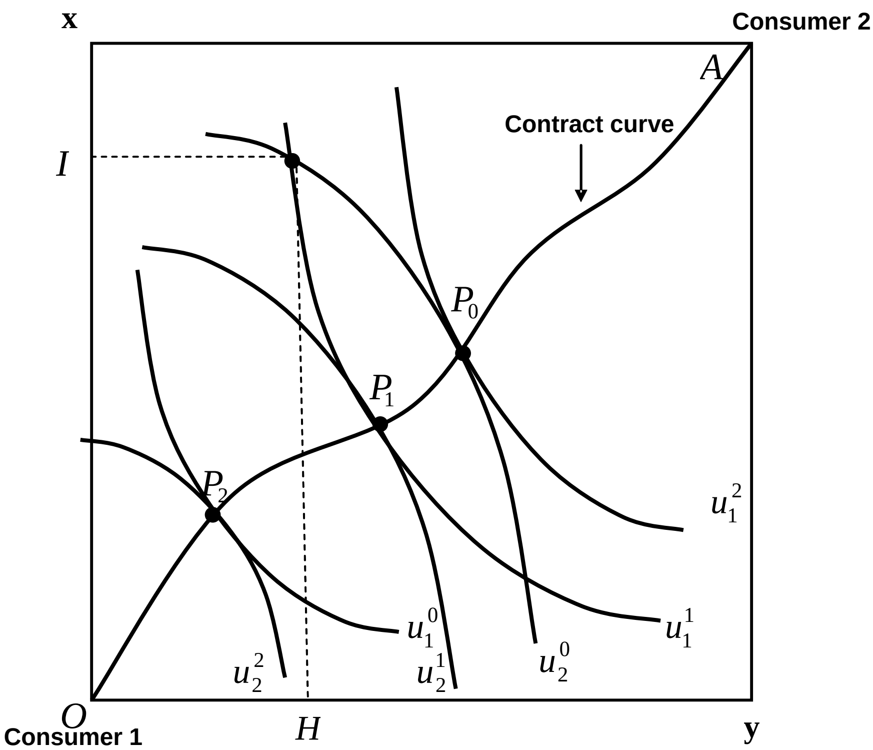

Figure 2: Pareto Optimal Allocations in the Edgeworth Box

Consumer 1's indifference curves are increasing in utility when one moves in the north eastern direction in the box, while Consumer 2 benefits from moves in the south west direction. The curve OA (the contract curve) is the locus of points where the Consumers' indifference curves are tangent to each other. This means that at a point on the curve, e.g., P_1, it is impossible to increase the utility of one consumer without decreasing the utility of the other. Given an initial endowment, i.e. the coordinates (H, I) where two indifference curve cross, it is obvious that the allocation can be improved for both individuals.

To prove a general version of the Second Welfare Theorem we need a more involved proof, and we have chosen to introduce a less general version that allows us use a brief proof. The condition we introduce that simplifies the proof is to assume that a Walrasian (competitive) equilibrium exists.

Theorem 2: Suppose $\mathbf{x}^*$ is a Pareto efficient allocation, that preferences are non-satiated, and that a Walrasian equilibrium exists from the initial endowment $\mathbf{z}_i = \mathbf{x}_i^*$, for all i. Denote this equilibrium $(\mathbf{p}', \mathbf{x}')$. Then $(\mathbf{p}', \mathbf{x}^*)$ is a Walrasian equilibrium.

Proof: Since x_i^* is feasible by construction, we know that $\mathbf{x}_i' R_i \mathbf{x}_i^*$ all i. Since x^* is Pareto efficient this implies $x_i' I_i x_i^*$. Thus both vectors provide maximum utility on the budget set. Hence, (p', x^*) is a Walrasian equilibrium.

Remark: Non-satiated means that you locally always can find a preferred consumption bundle, i.e., indifference curves are not thick lines.

A competitive equilibrium exists if excess demand functions are continuous. This can be fixed by assuming that preferences are strictly convex, continuous, and monotonic. Strict convexity means that if the consumer has the ranking $\mathbf{x_i} R_i \mathbf{y}_i$ and $\mathbf{x_i'} R \mathbf{y}_i$, then the vector $[t\mathbf{x_i} + (1-t)\mathbf{x_i'}] P_i \mathbf{y_i}$, for $0 < t < 1$. In other words, if both $\mathbf{x_i}$ and $\mathbf{x_i'}$ are preferred, or indifferent to $\mathbf{y_i}$, then all points along a straight line between $\mathbf{x_i}$ and $\mathbf{x_i'}$ are strictly preferred to $\mathbf{y_i}$. That the preferences are continuous means, loosely speaking, that if a sequence of consumption bundles $(\mathbf{x}_i)$ that all are preferred to a bundle $\mathbf{y_i}$ converges to a consumption bundle $\mathbf{x_i^*}$, then the latter is also preferred to $\mathbf{y_i}$. Finally, monotonicity means that if $\mathbf{x}_i \geq \mathbf{y}_i$, and, $\mathbf{x}_i \neq \mathbf{y}_i$, then $\mathbf{x}_i P \mathbf{y}_i$. Intuitively speaking, a little more of something is always better than status quo. Hence, Theorem 2 can be rephrased in the following manner:

Theorem 2a: Suppose $\mathbf{x}^*$ is a Pareto efficient allocation with $\mathbf{x}_\mathbf{i}^* >> 0$ all *i*, and preferences are convex, continuous and monotonic. Then x^* is a Walrasian equilibrium for the initial endowment $\mathbf{z}_i = \mathbf{x}_i^*$, all *i*.

The notation $\mathbf{x}_\mathbf{i} >> 0$ means that every component of the vector is greater than zero. A proof of Theorem 2a is available in Varian (1994) Chapter 13.

Note here that this means that an initial non-efficient endowment can be reallocated to fit any Pareto-optimum. Hence, all points along the contract curve inside the Edgeworth box in Figure 2 can be reached by lump sum transfers of the initial endowment and a suitable price vector.

To understand the mechanism involved we introduce Figure 3.

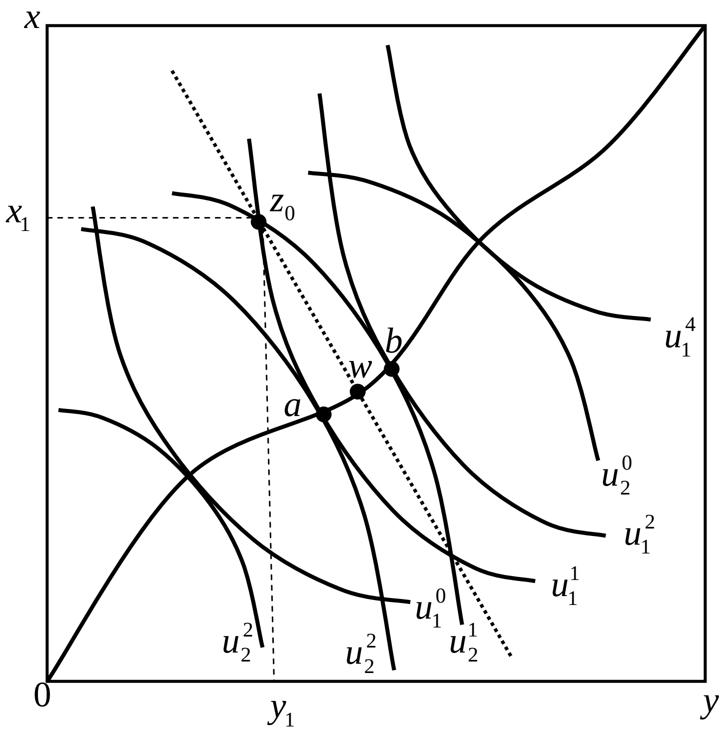

Figure 3: The Walrasian Equilibrium and the Core

Starting from the initial allocation, $\mathbf{z}_0$ in the Edgewoth box in Figure 3, there is, however, only a small part in the lens between the indifference curves A and B, where the Walrasian equilibrium can end up. This is the segment ab of the contract curve *AB*. Why is this? The reason is simple. Neither trader would accept trades that move them to a lower utility level. At point **w, (the Walrasian equilibrium)** on the contract curve all mutually advantageous trades are exhausted and the economy has reached a Pareto optimum. The slope of the line between $\mathbf{z}_0$ and **w** is the Walrasian equilibrium (relative) price vector. The segment ab of the contract curve is called the *Core* of the market economy, and in this case, two goods and two consumers, the Walrasian equilibrium belongs to the Core, and all points in the Core are Pareto efficient allocations. We now show that this is true in a more general context. We start by defining the Core in a rigorous manner.

Definition 8: A group of agents S (a coalition S) is said to improve upon a given allocation $\mathbf{x}$ if there is some allocation $\mathbf{x}'$ such that

(i) $\sum_{i \in S} \mathbf{x}'_i = \sum_{i \in S} \mathbf{z}_i$ ($\mathbf{x}'$ is feasible for S)

(ii) $\mathbf{x}'_i$ is preferred to $\mathbf{x}_i$ for all $i \in S$.

Now we use Definition 8 to define the Core of a market economy.

Definition 9: A feasible allocation $\mathbf{x}$ is in the Core of the economy if it cannot be improved upon by any coalition.

The allocations on the segment ab in Figure 3 all belong to the Core, since it cannot be improved upon by the grand coalition consisting of two agents, or the singleton coalitions at the initial endowment.

We can now prove the following theorem:

Theorem 3: If $(\mathbf{x}^*, \mathbf{p}^*)$ is a Walrasian equilibrium from an initial endowment $\mathbf{z}$, then the allocation $\mathbf{x}^*$ is in the Core.

Proof: Assume not; then there is some coalition S and some feasible allocation $\mathbf{x'}$ such that all agents in S prefer $\mathbf{x}'_i$ to $\mathbf{x}^*_i$ and

$$\sum_{i \in S} \mathbf{x}'_i = \sum_{i \in S} \mathbf{z_i}$$

The definition of a Walrasian equilibrium, $\mathbf{x}^*$, implies that $\mathbf{p}^*\mathbf{x}^*_i > \mathbf{p}^*\mathbf{z}_i$ all $i \in S$, i.e., it costs more than the initial endowment at general equilibrium prices. Summing yields $p^* \sum_{i \in s} x'_i > p^* \sum_{i \in S} \mathbf{z}_i$, violating feasibility. The contradiction proves the theorem.

5. Gains From Free Trade

That the Walrasian equilibrium belongs to the allocation that cannot be improved upon is based on two fundamental properties of a competitive market. The first is that the agents trade if and only if the trades are mutually advantageous. The second is that in a competitive equilibrium all mutually advantageous trades are exhausted. These properties may also help us understand that there are gains from free trade compared to autarky. In this section, we show, using tools from modern trade theory, that Robinson Cruse would have gained from free trade.

Let us start with some preliminaries. Assume that Robinson has a utility function $u(x, \bar{z} - l^s)$, where $\bar{z}$ is the fixed initial endowment of leisure time. The utility function is assumed to be increasing in consumption and leisure time. This is only a slight modification of the model in section 2. Here we are more explicit about leisure time, $\bar{z} - l^s$.

Instead of maximizing utility subject to a budget constraint like in section 2, we introduce a new concept, the expenditure function defined by the following minimization problem:

$$e(p, w, \bar{u}) = \min_{x, l^s} [px + w(\bar{z} - l^s)]$$

subject to (17)

$$u(x, \bar{z} - l^s) \geq \bar{u}$$

In words, the expenditure function gives us the minimum expenditure Robinson needs to reach utility level $\bar{u}$. Clearly, the higher the utility level the more income is needed to reach it. Technically speaking, the expenditure function is increasing in $\bar{u}$. It is also increasing in w and p.

Robinson is also a producer, and the revenues that are produced are maximized subject to the given production technology he uses, i.e.:

$$R(p,w,\bar{z}) = \max_{x^s,l}[px^s + w(\bar{z}-l)] = \max_{x^s,l}[px^s - wl + w\bar{z}]$$

subject to (18)

$$x^s = f(l)$$

The function $R(p,w,\bar{z})$ is called the revenue function, but note that the resulting supply of goods and the demand for labor coincide with the quantities that maximize profit, since $w\bar{z}$ is a constant. The revenue function is increasing in w and p. Moreover, from the Envelope Theorem that was introduced in section 3, we can derive the following results:

$$\begin{aligned}\frac{\partial e(\bullet)}{\partial p} &= x(p,w,\bar{u}) \\ \frac{\partial e(\bullet)}{\partial w} &= \bar{z} - l^s(p,w,\bar{u})\end{aligned} \qquad (19)$$

and

$$\begin{aligned}\frac{\partial R(\bullet)}{\partial p} &= x^s(p,w) \\ \frac{\partial R(\bullet)}{\partial w} &= \bar{z} - l(p,w)\end{aligned} \qquad (20)$$

Equations (19) are called the compensated (Hicksian) demand (supply) functions since they are conditioned on a given utility level.

A competitive equilibrium corresponding to the one in section 2 can now be defined by a vector (p^*, w^*, u^*) such that

$$R(p^*, w^*, \bar{z}) = e(p^*, w^*, u^*, \bar{z}) \qquad (21)$$

Equation (21) tells us that optimal (maximum) revenue equals optimal (minimum) cost. Therefore, the vector that maximizes profit helps us to reach the equilibrium utility level at minimum cost (efficiently).

We are now ready to prove the following theorem:

Theorem 4: Free trade can be no worse than autarky.

Proof: Suppose that autarky equilibrium is given by $c^a = (x_a, \bar{z} - l_a^s)$ and $u^a = u(c^a)$. Let $P^t = (p^t, w^t)$ be the equilibrium price vector under free trade, and u^t the corresponding utility level. By using the properties of the expenditure and revenue functions we can come up with the following series of weak inequalities: $e(P^t, u^a) \leq P^t c^a \leq R(P^t, T) = e(P^t, u^t)$. Now, $u^t \geq u^a$, since the expenditure function is increasing in u. This proves the claim.

Remarks: The first weak inequality follows by the definition of the expenditure function. It says that the autarky utility level can be obtained more economically at trade prices. The second weak inequality follows by definition of the revenue function, and the final part follows by the equality between the revenue and expenditure functions in equilibrium. The duality approach introduced in connection with Theorem 4 was popularized in the monograph by Dixit and Norman (1980)

There are also two other important points to be made. The first is that the price vector P^t must be different from the equilibrium price vector under autarky, P^a, to generate trade. The second is that Robinson has to be able to trade (sell or buy) labor and goods. If the two price vectors coincide, Robinson would remain satisfied with the autarky equilibrium. With respect to the second point, we know from Section 2 that there is only one relevant price in this simple model, the real wage rate, $\frac{w}{p}$. The two possibilities for trade are $(w/p)^a > (w/p)^t$ and $(w/p)^a < (w/p)^t$. In the first case, Robinson would like to produce more goods by importing cheap labor (Friday?), and then sell these goods in the international market. In this manner he would get more leisure time for himself for the same (or smaller) amount of goods than under autarky. In the second case, he would like to produce a smaller amount of goods at home than under autarky and sell his labor in the international market; furthermore, he would like to buy cheap goods in the international market. In the new equilibrium, he would have less leisure time and more goods than under autarky. Therefore, and depending on the compensation Friday wants, he might be able to do even better by sending him to the international market. However, as neither Robinson nor Friday can leave the island, both cases are impossible; they just illustrate how hampering for welfare barriers to trade might be.

Theorem 4 is valid for any number of goods, but adding more consumers creates distributional complications, which we would need some kind of welfare function to handle. We therefore turn to distributional issues and Arrow's Impossibility Theorem (or the Third Fundamental Welfare Theorem) to see if such a function exists.

6. Arrow's Impossibility Theorem

The Second Welfare theorem (Theorem 2 above) raises an important question: Which one of the many possible Pareto efficient allocations should be chosen? Or almost equivalently, what should the income distribution look like, and how should it be chosen? A commonly used method to choose between alternatives is majority voting, but it has long been known that it has an important flaw. The flaw is called the voting paradox. It has been known since Markis de Condorcet's published his *Essai sur l'Application de l'Analyse a' la Probablite' des Decisions Rendue a' la Pluralite' des Voix* in *1785*. It was also mentioned by the Auastralian E.J Nansen hundred years later (1882), and taken up by the political scientist Duncan Black in 1948.

The standard illustration of it assumes three individuals and three alternatives. The three voters have the following preferences:

(1) x y z

(2) y z x

(3) z x y

The alternatives are listed in the order they rank from left to right. Majority voting between each pair of alternatives shows that *x wins over y* and *y wins over z,* but surprisingly also *z wins over x.* Such voting cycles means that we cannot use a majority rule to solve the distributional problem. A possible alternative is to introduce a Bergsonian Welfare Function, where the individual utility functions are weighted arguments. “Evil people” say that this is a roundabout way to discover Pareto efficiency, or, if enough detail is added, a method to discover the constructor’s preferences. We will, nevertheless, use such welfare functions in later sections.

Arrow asks how individual preference orderings can be transferred into a social preference ordering. Technically, he asks for a mapping of individual orderings into a social ordering;

$$T(R_1, R_2, ..., R_n) \to R \qquad (22)$$

To be able to move beyond weak orderings like Pareto optimality, the social ordering R *must be able to rank all alternatives,* and to avoid cycles like the voting paradox it must *be transitive.* In other words, it must fulfill Axioms 1 and 2 in section 5, i.e. completeness and transitivity. But this is not enough; more structure has to be added to make the transformation in equation (22) reasonable for social choice.

Arrow (1951) introduced some reasonable constraints or conditions that the social preference ordering has to satisfy. Here we will use a modified (stronger) version of Arrow’s axioms, to be able to prove a simplified version of Arrows Impossibility Theorem. The following axioms, which were introduced by Sen (1970), constitute a condensed version of the conditions that were imposed by Arrow (1951):

(1) Universality (Completeness): The function T is a function of individual preferences and should always be able to rank alternatives independently of the shape of the preferences of the individuals.

(2) Pareto consistency: If all individuals prefer x to y society should prefer x to y

(3) Neutrality-Independence-Monotonicity: Assume all individuals in a set *S* prefer *x* to *y*, all individuals not in S prefer *y* to *x*, and society prefers *x* to *y*. Then society must prefer *x* to *y* for any preference profile that fulfills the condition that all individuals in *S* prefer *x* to *y*

(4) Non-dictatorship: There is no individual *i* such that xR_iy implies xRy independent of the preferences of all other individuals.

The first condition tells us that there can be no "holes" in the social preference ordering. Society must be able to rank any two alternatives. The second seems very reasonable indeed. If it does not hold, society's preferences could be imposed by outsiders. The third condition is not innocuous, and a strengthening of a condition that Arrow referred to as **Independence of irrelevant alternatives.** Loosely speaking, the latter means that if there are two preference orderings under which all individuals prefer *x* to y, then if society prefers x to y, it will do so independently of how individuals rank a third alternative z (for example in relation to *x* and *y)*. This assumption has been criticized, but the attempts to weaken it have also been criticized and not able to change the Impossibility Theorem into a consistent and credible welfare function.

It is worth noting that the market mechanism also operates independently of irrelevant alternatives. To see this, assume that we alter individual preferences for alternatives that are socially infeasible which, in turn, will not change the competitive equilibrium. Hence, a competitive equilibrium must, given the Impossibility Theorem, violate some other condition(s). In Arrow's original setting, it violates a condition that he calls "collective rationality" meaning that society's indifference curves are not allowed to cross.

We are now ready to prove a version of the Impossibility Theorem. The proof has been borrowed from Allan F. Feldman in an essay on Welfare Economics published in *The New Palgrave: A Dictionary of Economics,* 1988, Volume Q-Z. To that end we introduce the following definition:

Definition 10: A group of individuals, *S*, is called decisive whenever all people in *S* prefer *x* to *y*, society prefers *x* to *y*.

We can write the Impossibility Theorem as follows:

Theorem 5: There is no welfare function $T(R_1, R_2, ...R_n)$ that satisfies conditions (1)-(4).

Proof: We start by observing that there always exist a decisive group of individuals, since by the Pareto consistency condition the set of all individuals (society) is one. Let us now assume that *S* is a decisive set of individuals of minimal size. If *S* contains only one individual he is a dictator, and this violates Condition 4. Hence, we assume the decisive group *S* contains more than one individual and divide *S* into two non-empty sets S_1 and S_2. We denote by S_3 all individuals who are in neither S_1, nor S_2.

By Universality the function $T(\bullet)$ has to be applicable to any profile of individual preferences. Assume that the three group of voters have the following preferences:

For individuals in group S_1*:* $x \quad y \quad z$

For individuals in group S_2*:* $y \quad z \quad x$

For individuals in group S_3*:* $z \quad x \quad y$

Since *S* is decisive yPz. From Completeness either xRy or yPx. If xRy, yPz and transitivity implies that xPz. This means that S_1 is decisive, contradicting that *S* is the minimal decisive group.

If yPx, then S_2 is decisive, contradicting that S is the minimal decisive group. Hence, any attempt to avoid a dictator will result in contradiction. This means that S contains only one individual, and he is a dictator.

A lot of effort has been spent on attempts to weaken the conditions the social ordering must satisfy aiming at a "Possibility Theorem", but no unanimous conclusion has yet been reached. However, Duncan Black in his revival of the voting paradox in 1948 came up with an application that solved the problem in a special case. He was a Political Scientist and noted that political preferences are often single peaked; i.e. on a left right scale your preferences are such that of two parties to your left (right) you have the strongest preference for the party that is the one less to the left (right). More precisely, he assumes that if $U_{1,} U_2, ..., U_n$ are utility indicators for the individual orderings $R_1, R_2, ..., R_n$, then the alternative social states can be represented by a one-dimensional variable in such a way that each of the graphs of $U_1, U_2, ..., U_n$ has a single peak. Arrow (1951) mentions an economic example in terms of a working time leisure trade off. He assumes that, for reasons of technological efficiency, all workers work the same number of hours, and the more hours work the lower the marginal product of work and consequently the real wage rate. This means that each social alternative is given by a single number, the hours worked. Given the typical shape of indifference curves; the closer (on either side) the tradeoffs are to the optimum, the higher the utility. Hence, individual preferences are single peaked.

Black shows that under his assumption of single-peaked preferences, majority decision will avoid cycles, and there is exactly one alternative that will receive a majority over any other alternative, provided that the number of voters is odd. He assumes a finite number of alternatives, while Arrow shows that the result is true for any number of alternatives.

7. Externalities

So far, we have discussed the properties of a competitive equilibrium. We have seen (among other things) that a competitive equilibrium is a Pareto efficient resource allocation, and that a Pareto efficient resource allocation can be supported by competitive prices. This section extends the analysis to market failures, which will be exemplified by consumption externalities associated with environmental damage. Therefore, without appropriate policies to correct for this market failure, the decentralized resource allocation is no longer necessarily efficient.

Let us once again use the model set out above, while modifying it appropriately to capture the consequences of consumption externalities. Consider an economy with n identical individuals. The utility function facing each individual is written as

$$u = u(c, z, e) \tag{23}$$

where c and z refer to private consumption and leisure, respectively, while e measures the environmental damage, which is treated as exogenous by each individual. Leisure is defined as a time endowment less the time spent in market work, *l*. The function $u(\cdot)$ is increasing in *c* and *z*, decreasing in *e* and strictly quasiconcave. We also assume that $e = nc$, meaning that the aggregate consumption causes environmental damage.

As we are focusing on consumption externalities, which are generated by the consumption behavior, the production sector plays no important role in the analysis here. We will, therefore, simplify by considering a linear technology, where the wage rate is fixed. This means that the resource constraint for the economy as a whole (i.e. the constraint on the consumption possibilities) can be written as

$$nwl - nc = 0 \tag{24}$$

where *w* is the wage rate. In Section 10, where we consider a consumption externality generated by production, we return to the more general formulation of the production technology used before.

In this simple economy, in which all individuals are identical, it is natural to assume the social welfare function (which is denoted by *W* throughout this section) is utilitarian, so $W = nu$. Therefore, we can write the Lagrangean corresponding to the social optimization problem as

$$L = W + \gamma n[wl - c] \tag{25}$$

The first order conditions for *c* and *l* become

$$u_c(c, z, e) + u_e n - \gamma = 0 \tag{26}$$

$$-u_z(c, z, e) + \gamma w = 0 \tag{27}$$

Equations (24), (26) and (27) together with the relationship $e = nc$ define the socially optimal resource allocation, which is denoted $(c^*, z^*, e^*, \gamma^*)$.

However, the socially optimal resource allocation is not automatically achieved in the market economy. The reason is that each individual treats *e* as exogenous and, therefore, disregards the negative impact of his/her consumption on the utility of other individuals. In a decentralized economy, each individual chooses c and l to maximize the utility given by equation (23) subject to the budget constraint

$$wl - c(1+t) + R = 0 \tag{28}$$

where t is a consumption tax and R a lump-sum transfer, which are treated as exogenous by the individual. Note also that, as the only role of the public sector in this model is to internalize externalities, the public budget constraint becomes $tc=R$.

In an uncontrolled market economy, where t=0, the first order conditions for the individual read

$$u_c(c,z,e) - \gamma = 0 \tag{29}$$

$$-u_z(c,z,e) + \gamma w = 0 \tag{30}$$

By comparing equations (26) and (29), it is clear that the first order condition for private consumption in the uncontrolled market economy differs from its counterpart in the social optimization problem.

However, by reintroducing the consumption tax mentioned above, and repaying the tax revenues lump-sum to the consumer, the first order condition for consumption obeyed by each individual changes to read

$$u_c(c,z,e) - \gamma(1+t) = 0 \tag{31}$$

Therefore, by choosing this consumption tax in a particular way, we can reproduce the social first order conditions. To be more specific, if we were to choose

$$t = -\frac{u_e(c^*,z^*,e^*)n}{\gamma^*},$$

it is clear that equation (31) becomes equivalent to equation (26). In other words, the social optimum, i.e. (c^*,z^*,e^*,γ^*), obeys the first order conditions derived in the decentralized economy. This means that the externality has become fully internalized, and the tax used to reach the social optimum exemplifies the concept of 'Pigouvian' tax discussed in the introduction. We will return to externalities and Pigouvian taxes in Section 10 in the context of an intertemporal model.

8. Public Goods

Private goods are typically described as rival and excludable. Rivalry means that there is rivalry over consumption in the sense that the amount consumed by each individual cannot be consumed by anyone else, whereas excludability implies that the owner of a unit of the good can exclude others from enjoying the benefits of its consumption. A pure public good, on the other hand, is described as non-rival and non-excludable. We shall in this section briefly discuss public goods and, in particular, how the optimality condition for a public good depends on the tax instruments used to raise revenue. We start with the famous Samuleson Rule, which gives the optimality condition for provision of a public good in a first best framework, and then continue by discussing (some of) the consequences of distortionary finance. As a final concern, we address public good provision and redistribution simultaneously by introducing asymmetric information between the private sector and the government.

8.1 Public Provision in the First Best: The Samuelson Rule

We use a slightly modified version of the model examined in Section 7. As before, we assume that the production technology is linear (nothing essential is lost by this simplification), and that all consumers are identical. The latter assumption will be relaxed in subsection 8.3, where we also consider distributional aspects of public policy.

The utility function facing each individual is given by

$$u = u(c, z, g) \tag{32}$$

where c and z have the same interpretations as before, whereas g is a pure public good provided by the government (or social planner). The resource constraint changes to read

$$nwl - nc - \kappa g = 0 \tag{33}$$

where κ is interpretable as a fixed resource cost of providing a unit of the public good measured in terms of lost private consumption. In other words, κ measures the marginal rate of transformation between the public good and the private consumption good. Many earlier studies normalize κ to one; however, for illustrative purposes, and to be able to distinguish different components of the marginal cost of providing public goods in the next subsection, we use this explicit notation in what follows.

Once again, as the individuals are identical, we assume the social welfare function is utilitarian, so $W = nu$. The social optimization problem will be to maximize the social welfare function subject to the resource constraint given by equation (33). If γ is used to denote the Lagrange multiplier associated with the resource constraint, the additional first order conditions can be written as

$$u_c(c, z, g) - \gamma = 0 \tag{34}$$

$$-u_z(c, z, g) + \gamma w = 0 \tag{35}$$

$$n u_g(c, z, g) - \gamma\kappa = 0 \tag{36}$$

By combining equations (34) and (36), we can immediately deduce the following rule for the public good, which is typically referred to as the Samuelson Rule (Samuelson (1954));

$$n\, MRS_{g,c} = \kappa \tag{37}$$

where $MRS_{g,c} = u_g(c, z, g) / u_c(c, z, g)$. The Samuelson Rule is a first best policy rule for public provision; it presupposes that the government or social planner can collect revenue by using lump-sum taxes. This condition would take the same form in an economy with heterogeneous consumers, provided that the government can use individual-specific lump-sum taxes. It means that the public good should be provided up to a point where the sum of marginal rates of substitution between the public good and the private consumption good equals the marginal rate of transformation between these goods. The sum of marginal rates of substitution is interpretable in terms of the sum of marginal willingness to pay for the public goods by the beneficiaries (note that all individuals benefit from the public good by assumption), measured in terms of lost private consumption, whereas the marginal rate of transformation is the marginal production cost.

So far, we have assumed that the public good is provided by the government. Is it possible to design a 'price system', such that the consumers, themselves, would choose an efficient amount of the public good in the context of a market? Without government intervention, the economy would most likely end up with inefficiently low public good provision. The reason is that each individual does not necessarily consider that his/her choice to contribute to the public good also affects the utility of others. Therefore, to reproduce equation (37), the government would have to subsidize the consumers. These subsidies are, in turn, related to the concept of Lindahl prices; see Lindahl (1919) and endnote 8. To illustrate, let us write the budget constraint facing each consumer as

$$wl - T - c - (1-s)\kappa\breve{g} = 0$$

where T is a lump-sum tax, $\breve{g}$ the individual contribution and s the subsidy rate. Therefore, budget balance from the perspective of the government implies $T = s\kappa\breve{g}$. Suppose that we were to choose[8]

$$s = \frac{(n-1)MRS^*_{g,c}}{\kappa},$$

where $MRS^*_{g,c} = u_g(c^*, z^*, g^*)/u_c(c^*, z^*, g^*)$ is defined by equation (37), which is part of the outcome of the first best social planner problem discussed above. Then, if each individual were to choose c, l and $\breve{g}$ to maximize utility subject to the hypothetical budget constraint given above, while treating the contributions by others as exogenous, it is easy to show that the first order conditions are given by equations (34), (35) and (36).

However, the reader should bear in mind that a large step still remains between theory and application; eliciting the willingness to pay for a public good is by no means a trivial empirical problem. The reason is, of course, that if the individuals were asked to state their willingness to pay, each of them may have incentives report a value that differs from the true willingness to pay. A mechanism that is designed to elicit the true willingness to pay – known as the Clarke-Groves Mechanism (Clarke (1971), Groves (1973)) – means that a system of side-payments is introduced to the individual alongside the willingness to pay question. To be more specific, each individual is asked to report a bid measuring his/her willingness to contribute to (i.e. willingness to pay for) the public good, and the individual is, at the same time, informed that he/she will receive a payment equal to the sum of the other bids if the public good is provided (if this sum is positive, the individual receives it; if it is negative, the individual must pay this amount). The system of side-payments implies that the level of the public good preferred by the individual is governed by the sum of his/her own willingness to pay and the other bids. Therefore, an interesting property of the side-payment is that, if all agents were to report bids that coincide with the true willingness to pay, the benefit of the public good perceived by each individual would be equal to the benefit for the society as a whole. The contribution of the Clarke-Groves Mechanism is then to ensure that reporting the correct willingness to pay is a dominant strategy for each individual, irrespective of what the others report[9].

8.2 Distortionary Revenue Collection

A long time ago, Pigou (1947) claimed that the rule for public provision summarized by equation (37) – which only recognizes the direct marginal benefit and production cost - does not apply if the public revenues are raised by distortionary taxes. The argument is simple; in an economy with distortionary taxes, revenue collection is, itself, costly, and this cost ought to be recognized also in the provision of public goods. Pigou wrote

"Where there is indirect damage, it ought to be added to the direct loss of satisfaction involved in the withdrawal of the marginal unit of resources by taxation, before this is balanced against the satisfaction yielded by the marginal expenditure. It follows that, in general, expenditures ought not to be carried so far as to make the real yield of the last unit of resources expended by the government equal to the real yield of the last unit left in the hand of the representative citizen" (1947, page 34).

This argument has been discussed by several prominent researchers such as Dasgupta and Stiglitz (1971) and Atkinson and Stern (1974). We will here concentrate on the first part of Pigou's statement, i.e. that the total marginal cost (including 'indirect damage') ought to be balanced against the marginal benefit. This means that our concern is to analyze the cost benefit rule for public provision and compare it to the first best policy rule discussed in subsection 8.1. To illustrate, let us write the budget constraint facing the representative consumer as

$$wl(1-\tau)-c=0 \tag{38}$$

where τ is the labor income tax rate. Each individual maximizes the utility function in equation (32) subject to the budget constraint in equation (38). The first order conditions for consumption and hours of work can be combined to imply

$$u_c(wl(1-\tau),z,g)w(1-\tau)-u_z(wl(1-\tau),z,g)=0 \tag{39}$$

By using $z=\overline{z}-l$, where $\overline{z}$ is a time endowment, equation (39) implicitly defines a labor supply function, $l=l(\omega,g)$, where $\omega=w(1-\tau)$ is the marginal wage rate. We can solve for the private consumption by substituting the labor supply function into the private budget constraint.

Suppose once again that a benevolent government with a utilitarian objective provides the public good. The decision-problem facing the government is to choose τ and g to maximize the social welfare function

$$W=nv(\omega,g)=nu(c(\omega,g),\overline{z}-l(\omega,g),g) \tag{40}$$

subject to its budget constraint

$$\tau nwl(\omega,g)-\kappa g=0 \tag{41}$$

Note that, in formulating the social welfare function, we have made use of the indirect utility function, $v(\cdot)$, as the labor supply and consumption behavior of each individual is recognized by the government. Let γ denote the Lagrange multiplier associated with the government budget constraint. In addition to equation (41), the first order conditions then become

$$-n\,wv_{\omega} + \gamma n[wl - \tau w^2 l_{\omega}] = 0 \qquad (42)$$

$$n\,v_g + \gamma[\tau nwl_g - \kappa] = 0 \qquad (43)$$

in which subindices denote partial derivatives. To shorten the notations, we have suppressed the arguments in the functions $v(\cdot)$ and $l(\cdot)$ in equations (42) and (43). Let us now analyze what these conditions imply in terms of public good provision. According to the Envelope Theorem, $v_g(\omega, g) = u_g(c, z, g)$, which enables us to write equation (43) in the same form as equation (37)

$$n\,MRS_{g,c} = m[\kappa - n\tau wl_g] \qquad (44)$$

where

$$m = \frac{\gamma}{u_c(c, z, g)}$$

is typically referred to as the marginal cost of public funds. This term was equal to one in the preceding subsection, where we used lump-sum taxes to collect revenue. One interpretation of the marginal cost of public funds is that it represents the marginal cost of raising tax revenues (measured in real terms), whereas another equivalent interpretation is that it is the multiplier to be applied to the direct marginal resource cost of the public good to provide the correct incentives for public good provision. The latter will be apparent in the special case with a separable utility function to be discussed below. By using the Envelope Theorem once again to show that $v_{\omega}(\omega, g) = u_c(c, z, g)l$, and if the labor supply is upward sloping, so $l_{\omega} > 0$ (which appears to be a reasonable assumption), we see that equation (42) implies $\gamma > u_c(c, z, g)$, so $m > 1$. Let us rewrite equation (42) to read

$$m = \frac{1}{1 - \dfrac{\tau}{1-\tau}\varepsilon} \qquad (45)$$

where $\varepsilon = l_{\omega}\omega / l$ is the hours of work elasticity with respect to the marginal wage rate. According to equation (45), the marginal cost of public funds depends on two important determinants; the income tax rate and the labor supply elasticity with respect to the marginal wage rate. An increase in either of these variables contributes to increase the right hand side of equation (45). Combining equations (44) and (45), we see that Pigou's conjecture is correct in part. Indeed, $m > 1$ works to increase the right hand side of equation (44) – relative to the case where $m = 1$ - which, in turn, necessitates a modification of the public provision to increase the sum of marginal rates of substitution (an adjustment which one would normally expect to lead to less public provision)[10]. However, the public good will, itself, also affect the tax revenues, which is seen from the second term within the square bracket on the right hand side of equation (44), and this effect can be either positive or negative depending on whether leisure is complementary with, or substitutable for, the public good.

If the public good is separable from the other goods in the utility function, meaning that the hours of work no longer directly depend on the provision of the public good, equation (44) simplifies to read

$$n\, MRS_{g,c} = m\, \kappa \tag{46}$$

Equation (46) means that the sum of marginal rates of substitution between the public good and the private consumption is equal to the product of the marginal rate of transformation and the marginal cost of public funds, which accords very well with the first part of the statement by Pigou mentioned above.

For further analysis of public good provision, the reader is referred to Wilson (1991), Sandmo (1998) and Gaube (2000). Gaube gives a thorough analysis of the second part of Pigou's statement; namely, whether the use of distortionary taxes leads to less provision of public goods compared to the situation with lump-sum taxes.

8.3 Briefly on Heterogeneity and Asymmetric Information

Although the analysis in the previous subsection exemplifies why distortionary taxation may necessitate a modification of the Samuelson rule for public good provision, it is by no means a realistic description of real world public policy. This is so for at least two reasons. First, there is no apparent reason for the government to use distortionary taxation, other than that we have assumed that it must do so. Rather, if all individuals are identical, one would expect that lump-sum taxes are feasible. Second, we have not addressed redistribution (which is, of course, also a consequence of using a representative agent economy). We will here briefly address public good provision in an economy, where the government redistributes by using nonlinear taxes.

There is a large and growing literature dealing with public policy in economies with nonlinear tax instruments[11]. Why is it interesting to extend the study of public good provision to an economy with nonlinear taxes? One reason is that nonlinear income taxes constitute a reasonably realistic description of the tax instruments that many countries have (or potentially have) at their disposal. In this case, the decision made by the government to use distortionary income taxation will follow from optimization, subject to the available information, and not be a direct consequence of arbitrary restrictions imposed on the set of policy instruments. Another reason is that this framework also sheds further light on the interesting question of why it is optimal to deviate from the first best Samuelson rule.

We follow Boadway and Keen (1993), who base their analysis on the two-type optimal income tax model developed by Stern (1982) and Stiglitz (1982). Therefore, consider an economy with two ability-types; a low-ability type (denoted by superindex 1) and a high-ability type (denoted by superindex 2). This distinction refers to productivity, meaning that the high-ability type faces a higher before tax wage rate than the low-ability type. Without any loss of generality, we normalize the number of individuals of each ability-type to one. The decision-problem facing ability-type *i* is written as

$$\underset{c^i, l^i}{Max} \; u(c^i, z^i, g)$$

subject to

$$w^i l^i - T(w^i l^i) - c^i = 0$$

where $T(\cdot)$ is the income tax payment. As before, we assume that each individual treats the public good as exogenous. The first order condition for the hours of work can be written as

$$u_c^i w^i [1 - T'(w^i l^i)] - u_z^i = 0 \tag{47}$$

where $T'(\cdot)$ is the marginal income tax rate, and we have used the short notation $u^i = u(c^i, z^i, g)$. Subindices denote partial derivatives.

We assume the government faces a general social welfare function

$$W = W(u^1, u^2) \tag{48}$$

in which it attaches weight to the utilities of both ability-types. An alternative formulation would be to assume that the government maximizes the utility of one of the ability-types subject to a minimum utility restriction for the other. This formulation would give the same expressions for the marginal income tax rates and policy rule for public provision as those derived below.

As in most of the earlier literature on the self-selection approach to optimal taxation, the government can observe income, although ability is private information. We also assume that the government wants to redistribute from the high-ability to the low-ability type. Therefore, we would like to prevent the high-ability type from mimicking the low-ability type in order to gain from redistribution. In other words, the high-ability type should at least weakly prefer the allocation intended for him/her over the allocation intended for the low-ability type. A high-ability type who pretends to be a low-ability type is called a mimicker. The self-selection constraint that may bind then becomes

$$u^2 = u(c^2, z^2, g) \geq u(c^1, H - \phi l^1, g) = \hat{u}^2 \tag{49}$$

where $\phi = w^1 / w^2$ is the wage ratio (relative wage rate), implying that ϕl^1 is the hours of work that the mimicker must supply to reach the same income as the low-ability type. The expression on the right hand side of the weak inequality is the utility of the mimicker, which is denoted by the hat. The mimicker enjoys the same consumption as the low-ability type, although the mimicker enjoys more leisure (as the mimicker is more productive than the low-ability type).

Note that the function $T(\cdot)$ is a general labor income tax. As such, it can be used here to implement any desired combination of l^1, c^1, l^2 and c^2. The reader may, for instance, think of a tax function with ability-type specific intercepts and slopes. It is, therefore, convenient to directly use l^1, c^1, l^2 and c^2 as decision-variables, instead of choosing these variables indirectly via the tax function. As will be explained below, we can, nevertheless, infer what these choices imply in terms of tax policy. We write the budget constraint facing the government as

$$\sum_i [w^i l^i - c^i] - \kappa g = 0 \tag{50}$$

in which we have used $T(w^i l^i) = w^i l^i - c^i$.

The decision-problem facing the government is to choose l^1, c^1, l^2, c^2 and g to maximize the social welfare function given by equation (48) subject to the self-selection and budget constraints in equations (49) and (50). The Lagrangean is written

$$L = W(u^1, u^2) + \lambda[u^2 - \hat{u}^2] + \gamma[\sum_i (w^i l^i - c^i) - \kappa g] \tag{51}$$

The first order conditions are

$$-\frac{\partial W}{\partial u^1} u_z^1 + \lambda \hat{u}_z^2 \phi + \gamma w^1 = 0 \tag{52}$$

$$\frac{\partial W}{\partial u^1} u_c^1 - \lambda \hat{u}_c^2 - \gamma = 0 \tag{53}$$

$$-[\frac{\partial W}{\partial u^2} + \lambda] u_z^2 + \gamma w^2 = 0 \tag{54}$$

$$[\frac{\partial W}{\partial u^2}+\lambda]u_c^2-\gamma=0 \tag{55}$$

$$\sum_{i=1}^{2}\frac{\partial W}{\partial u^i}u_g^i+\lambda[u_g^2-\hat{u}_g^2]-\gamma\kappa=0 \tag{56}$$

where (as before) subindices denote partial derivatives, e.g. $u_c^i=\partial u^i/\partial c^i$ and similarly for the other variables.

Let us begin by briefly discussing the optimal tax structure and then turn to public good provision, which is the main issue in this section. Denote by $MRS_{z,c}^i=u_z^i/u_c^i$ the marginal rate of substitution between leisure and private consumption for ability-type *i*. By combining equations (52) and (53), we obtain

$$MRS_{z,c}^1[\lambda\hat{u}_c^2+\gamma]=\lambda\hat{u}_z^2\phi+\gamma w^1=0 \tag{57}$$

Now, using the private first order condition to derive $T'(w^1l^1)w^1=w^1-MRS_{z,c}^1$ and then substituting into equation (57), we can derive an expression for the marginal income tax rate of the low-ability type

$$T'(w^1l^1)=\frac{\breve{\lambda}}{w^1}[MRS_{z,c}^1-M\hat{R}S_{z,c}^2\phi] \tag{58}$$

in which $\breve{\lambda}=\lambda\hat{u}_c^2/\gamma$, whereas $M\hat{R}S_{z,c}^2=\hat{u}_z^2/\hat{u}_c^2$ is the marginal rate of substitution between leisure and private consumption facing the mimicker. Note that the mimicker needs to forego less leisure than the low-ability type to accomplish a given increase in the private consumption. Therefore, as pointed out by e.g. Stiglitz (1982), the expression within the square bracket is positive, implying $T'(w^1l^1)>0$. The intuition is that, by imposing a positive marginal income tax rate on the low-ability type (instead of a zero rate), mimicking becomes less attractive, which contributes to relax the self-selection constraint. In other words, the positive marginal income tax rate facing the low-ability type serves the purpose of discouraging mimicking; as such, it also creates additional room for redistribution. By using equations (54) and (55), and performing the same calculations for the high-ability type, we have

$$T'(w^2 l^2) = 0 \tag{59}$$

which is the 'two ability-type version' of the zero-at-the-top-result; e.g. Phelps (1973) and Sadka (1976). The intuition is that we cannot relax the self-selection constraint further by distorting the labor supply behavior of the high-ability type. However, note that the tax formulas derived above presuppose fixed gross wage rates, which we have assumed here. If the gross wage rates are endogenous, then the wage ratio (which, in part, determines the hours of work that the mimicker needs to supply to reach the same income as the low-ability type) is also endogenous. In this case (and if we assume constant returns to scale in production), one would typically find that the marginal income tax rate facing the high-ability type is negative, whereas the marginal income tax rate facing the low-ability type is positive[12]; see Stern (1982) and Stiglitz (1982).

Turning to public good provision, our concern is again to see how and why the second best policy rule for the public good deviates from the Samuelson rule. Note that the first term on the left hand side of equation (56) can be written as $\sum_i MRS^i_{g,c}(\partial W/\partial u^i)u^i_c$. Therefore, by solving equations (53) and (55) for $(\partial W/\partial u^1)u^1_c$ and $(\partial W/\partial u^2)u^2_c$, respectively, substituting the resulting expressions into equation (56) and rearranging, we obtain

$$\sum_i MRS^i_{g,c} + \breve{\lambda}[MRS^1_{g,c} - M\hat{R}S^2_{g,c}] = \kappa \tag{60}$$

where $M\hat{R}S^2_{g,c} = \hat{u}^2_g / \hat{u}^2_c$ is the marginal rate of substitution between the public good and private consumption for the mimicker. By analogy to the marginal income tax rates discussed above, the self-selection constraint again determines to what extent it is optimal to deviate from the first best policy rule. If leisure is complementary with public consumption in the sense that the marginal willingness to pay for the public good (measured as marginal rate of substitution between the public good and the private consumption good) increases with the use of leisure, then $MRS^1_{g,c} < M\hat{R}S^2_{g,c}$. In this case, equation (60) implies that we ought to supply the public good up to a point where the sum of marginal rates of substitution exceeds the marginal rate of transformation. If, on the other hand, leisure is substitutable for the public good in the analogous sense, so $MRS^1_{g,c} > M\hat{R}S^2_{g,c}$, equation (60) gives the opposite result: we ought to supply the public good up to a point where the sum of marginal rates of substitution falls short of the marginal rate of transformation.

The intuition behind these results is straight forward: we are able to use the public good as a means to relax the self-selection constraint and, therefore, create further room for redistribution. This provides an incentive for the government to deviate from the Samuelson rule. If leisure is complementary with the public good, implying that the mimicker attaches a higher marginal value to the public good than does the low-ability type, reducing the public provision below the level indicated by the Samuelson rule causes a greater utility loss for the mimicker than it does for the low-ability type. Therefore, reduced public provision enables us to discourage mimicking and relax the self-selection constraint. The intuition will be analogous in the case where leisure is substitutable for the public good.

This discussion also enables us to identify an interesting special case. If leisure is weakly separable from the other goods in the utility function, meaning that the utility function takes the form $u^i = \varphi(b(c^i, g), z^i)$, where $b(\cdot)$ is a subutility function defined over the private consumption good and the public good, then the marginal rate of substitution between the public good and the private consumption does not depend on the leisure choice (other than via income). In this case, therefore, $MRS^1_{g,c} = M\hat{R}S^2_{g,c}$, meaning that the second term on the left hand side of equation (60) is zero. As a consequence, the policy rule for the public good corresponds to the Samuelson rule; see also Christiansen (1981). The intuition is that we cannot, in this case, relax the self-selection constraint by deviating from the Samuelson rule, implying that the government in our model will not do so.

For additional reading on public good provision and/or other resource allocation problems in economies with nonlinear income taxes, see e.g. Mirrlees (1971), Phelps (1973), Sadka (1976), Atkinson and Stiglitz (1976), Christiansen (1981), Stern (1982), Stiglitz (1982), Boadway and Keen (1993), Edwards et al. (1994), Cremer and Ghavari (2000), and Aronsson and Sjögren (2004).

9. More on Mechanism design

The self selection constraint that was used in connection with optimal taxation, and the Clark-Groves mechanism which was introduced to elicit the willingness to pay for public goods, are two examples of a quite modern branch of Economics that is called Mechanism design (the basic idea behind the Clarke–Groves mechanism was already present in Vickery (1961); therefore, it is sometimes referred to as the Vickery-Clarke-Groves mechanism). Another example is Arrow's attempt to aggregate individual preferences into a social ordering that has certain desirable properties. Loosely speaking, he shows that there is no such mechanism. Other well known mechanism design principles are found in terms of auction mechanisms.

Mechanisms design was born from the discussions on planned versus market economies during the 1930s and 1940s, between Oscar Lange and Abba P Lerner on one side (planning), and Friedrich von Hayek and Ludwig von Mieses on the other (the market). Enrico Barone (1912) had suggested that a perfect market economy and a perfectly planned market economy were equivalent. This claim was supported by Lange and Lerner. Lerner even bought a car and went down to Mexico to convince a Trotsky in exile about the marginal principles. Mieses and Hayek argued that perfect markets had properties that could not be replicated by a planned system.

However, at the time, the conceptual apparatus needed to unambiguously define the competing claims, and the technical tools - notably game theory - to derive clear conclusions were lacking. Leonid Hurwicz (1960, 1972) led the way in developing both the conceptual framework and the theoretical tools. He coined the term "incentive compatibility". A mechanism is incentive compatible, if no agent has the incentive to pretend having any characteristics other than the true ones. Hence, the design of the optimal taxation problem is an example of an incentive compatible mechanism, and so is the Clarke-Groves mechanism. In particular Hurwicz pointed out that a market run by a Walrasian auctioneer would typically not satisfy incentive compatibility, if there is a finite number of individuals, because each buyer would understate her demand at a given price, and each seller would understate his supply (a continuum of individuals would help). He went on showing that no (privacy respecting) incentive compatibility mechanism can be Pareto efficient in the usual sense.

There remained a couple of important problems to be solved in a practically feasible manner. One was that to find an *optimal* mechanism under a given goal function, it is necessary to define the set of all feasible communication systems. However, such a set is very likely to be huge, and quite hopeless to represent and analyze. One breakthrough came with the development of “The Revelation Principle” (Allan Gibbard (1973), Partha Dasgupta, Peter Hammond, and Eric Maskin (1979), Robert Myerson (1979)). This principle says that one can restrict the search to a much smaller class called *direct mechanisms.* Under a direct mechanism, agents’ messages are reports (truthful or not) of their private information (type); in the optimal taxation problem above, whether they are high-skilled or low-skilled. A direct mechanism assigns outcomes to the list of messages, one from each agent. The mechanism is said to be incentive compatible, if it induces each individual to tell the truth about his/her type (private information). The revelation principle states that for any arbitrary mechanism, there exists a direct mechanism that does just as well in terms of the value of the objective (welfare) function. In other words, the mechanism design problem can be reduced to the choice of a direct mechanism that maximizes the value of the objective function, subject to an incentive compatibility constraint and a participation constraint. The latter sees to that an agent who has a possibility to exit the “game” prefers to stay. Typically it means that the outcome of the game gives the agents more than a minimum (reservation) utility level.

A second problem is, however, that incentive compatible mechanisms may give rise to multiple equilibria, of which at least one means truth telling, whereas some others do not. It would be desirable to design mechanism such that all equilibrium outcomes are (second best) social optima. Fulfilling the quest that every social optimum is an equilibrium outcome, and that every equilibrium outcome is a social optimum is an important implementation problem. This problem was “solved” by Eric Maskin (1977). He introduced a new class of mechanisms called *canonical mechanisms.* They are quite complex and cannot be interpreted as if agents only report their type. However, the good news is that Maskin’s results are “possibility theorems”, showing that under certain conditions, implementation is possible. The second best mechanisms cannot, however, get rid of Arrow’s dictator.

There are many applications of the mechanism design theory and the revelation principle. Many of them are connected to optimal auctions; others refer to public goods production, regulation and auditing and optimal procurement mechanisms. For Vickery-Clark-Groves mechanisms that have been used to solve practical problems, the reader is referred to Edward Clarke's homepage (www.clarke.pair.com/apppubgoods.html).

10. National Welfare Measures in Dynamic Economies

During the last 30 years, a theory of social accounting, which is based on growth theoretical models, has gradually been developed. One of the most interesting ideas behind the theory of social accounting is the aim of constructing a comprehensive net national product (NNP) measure, which can be used as a welfare indicator in a dynamic economy. The comprehensive NNP can be thought of as an extension of the conventional NNP, where the extension is designed to reflect all relevant aspects of consumption and capital formation for society. As the study of economy-wide welfare measures has become an increasingly important part of Welfare Economics, we will briefly discuss the theory of social accounting here. We start with the seminal contribution by Weitzman (1976) and then continue by explaining some of the more recent research developments.

10.1 A Basic Dynamic Model

In this subsection, we describe the model, whereas the formal welfare analysis is carried out in subsections 10.2 and 10.3 below. The model is based on Brock (1977), where production releases emissions. These emissions add to a stock of pollution, which causes a consumption externality. From our perspective, this model is suitable to use here for at least two reasons. First, by introducing a market failure, we are able to make a distinction between a first best welfare measure and a welfare measure applicable in an imperfect market economy. As we will argue below, this distinction is important for understanding the welfare foundation of comprehensive NNP. It also provides a framework for analyzing dynamic analogues to the Pigouvian taxes discussed in Section 7. Second, environmental aspects are often emphasized in the study of social accounting, meaning that we able to connect this section to a major theme in earlier literature; namely, how to make the national accounts 'greener'. Subsections 10.2 and 10.3 are largely based on Aronsson et al. (2004).

The consumers are assumed to be identical and have infinite planning horizons. We follow the convention in the literature on social accounting by disregarding population growth, and we normalize the population to equal one. In addition, as the labor supply behavior of the consumer is of no direct importance for the results to be derived below, we simplify by disregarding the utility of leisure and, instead, assume that the consumer supplies one unit of labor inelastically at each instant. The instantaneous utility function at time *t* is written as

$$u(t) = u(c(t), x(t)) \tag{61}$$

where *c* is private consumption (as before) and *x* the stock of pollution. We assume the function $u(\cdot)$ is increasing in *c*, decreasing in *x* and strictly concave.

Turning to the production side, we assume identical competitive firms, whose number is normalized to one, produce a homogenous good by using labor (normalized to one and suppressed), capital and emissions (through the use of energy inputs). The production function is given by

$$y(t) = f(k(t), g(t)) \tag{62}$$

where *y* denotes net output, meaning that depreciation has been accounted for, *k* the capital stock and *g* energy use. We assume that the function $f(\cdot)$ is increasing in each argument and strictly concave.

The stock of pollution accumulates according to the differential equation

$$\dot{x}(t) = g(t) - \gamma x(t) \tag{63}$$

where $\gamma \in (0,1)$ reflects the assimilative capacity of the environment. To connect emissions to energy input in a simple way, we assume (with very little loss of generality) that the emissions equal the input of energy.

The accumulation of physical capital obeys the differential equation

$$\dot{k}(t) = f(k(t), g(t)) - c(t) \tag{64}$$

10.2 Welfare Measurement in the First Best Social Optimum

To derive the first best social optimum, it is convenient to assume that the resource allocation is decided upon by a benevolent social planner, whose objective coincides with the utility function facing the representative consumer (recall that the consumers are identical, meaning that redistribution is not an issue). The decision-problem facing the social planner can be written as

$$\underset{c(t),g(t)}{Max}\int_0^\infty u(c(t),x(t))e^{-\theta t}dt \tag{65}$$

subject to

$$\dot{k}(t) = f(k(t),g(t)) - c(t) \tag{66}$$

$$\dot{x}(t) = g(t) - \gamma x(t) \tag{67}$$

as well as subject to initial conditions, $k(0) = k_0 > 0$ and $x(0) = x_0 > 0$, and terminal conditions, $\lim_{t\to\infty} k(t) \geq 0$ and $\lim_{t\to\infty} x(t) \geq 0$. The parameter θ is the intertemporal rate of time preference (or utility discount rate).

Neglecting the time indicator for notational convenience, the present value Hamiltonian can be written as

$$H = u(c,x)e^{-\theta t} + \lambda\dot{k} + \mu\dot{x} \tag{68}$$

where λ and μ are costate variables. In addition to equations (66) and (67), as well as to the initial and terminal conditions, the necessary conditions are (for more detail the reader is referred to Seierstad & Sydsaeter (1987), Theorem 3.16)

$$\frac{\partial H}{\partial c} = u_c(c,x)e^{-\theta t} - \lambda = 0 \tag{69}$$

$$\frac{\partial H}{\partial g} = \lambda f_g(k,g) + \mu = 0 \tag{70}$$

$$\dot{\lambda} = -\lambda f_k(k,g) \tag{71}$$

$$\dot{\mu} = -u_x(c,x)e^{-\theta t} + \mu\gamma \tag{72}$$

$$\lim_{t\to\infty} \lambda \geq 0 \,(= 0 \text{ if } \lim_{t\to\infty} k(t) > 0) \tag{73}$$

$$\lim_{t\to\infty} \mu \geq 0 \,(= 0 \text{ if } \lim_{t\to\infty} x(t) > 0) \tag{74}$$

where the subindices attached to the utility and production functions denote partial derivatives. Let

$$\{c^*(t), g^*(t), k^*(t), x^*(t), \lambda^*(t), \mu^*(t)\}_0^\infty$$

denote the socially optimal resource allocation. By totally differentiating the present value Hamiltonian with respect to time and using the necessary conditions, we have

$$\frac{dH^*(t)}{dt} = -\theta u(c^*(t), x^*(t))e^{-\theta t} \tag{75}$$

Equation (75) is a direct consequence of the dynamic Envelope Theorem: all indirect effects of time via control, state and costate variables vanish as a consequence of optimization, as the resource allocation obeys the necessary conditions given by equations (69)-(74). Therefore, only the direct effect of time remains in equation (75), which is due to the explicit time-dependence of the utility discount factor. By solving equation (75) subject to the transversality condition $\lim_{t\to\infty} H^*(t) = 0$, and transforming the solution to current value (multiplying by $e^{\theta t}$), we have[13]

$$\theta \int_t^\infty u(c^*(s), x^*(s))e^{-\theta(s-t)}ds = H^{c^*}(t) \tag{76}$$

in which $H^c = He^{\theta t}$ is the current value Hamiltonian. Although we have chosen to carry out the analysis in a utility metric (which is a choice motivated by convenience), an alternative is, of course, to use a money metric. See Li and Löfgren (2002).

Equation (76) is Weitzman's (1976) result applied to our model. It means that the present value of future utility at time *t* is proportional to the current value Hamiltonian at time *t*. The current value Hamiltonian is, in turn, interpretable as the comprehensive NNP in utility terms. To see this, note that the current value Hamiltonian can be written as

$$H^{c^*}(t) = u(c^*(t), x^*(t)) + \lambda^{c^*}(t)\dot{k}^*(t) + \mu^{c^*}(t)\dot{x}^*(t) \tag{77}$$

where $\lambda^c = \lambda e^{\theta t}$ and $\mu^c = \mu e^{\theta t}$ are current value costate variables. The current value Hamiltonian measures the utility value of the current consumption plus the utility value of the current net investments. For the simple economy considered here, the consumption concept refers to goods and services, *c*, and pollution, *x*, whereas the net investments refer to the changes in the physical capital stock, $\dot{k}$, and the additions to the stock of pollution, $\dot{x}$. To facilitate the interpretation of the current value Hamiltonian in terms of comprehensive NNP, let us linearize equation (77). By using equation (69), we can rewrite the instantaneous utility function as follows;

$$u(c,x) = \lambda^c[c + \rho x] + s \tag{78}$$

where $s = u(c,x) - \lambda^c c - u_x(c,x)x$ is the consumer surplus and $\rho = u_x(c,x)/u_c(c,x)$ the marginal rate of substitution between pollution and private consumption. For a thorough analysis of the role of the consumer surplus in social accounting, see Li and Löfgren (2002)[14].

We can now rewrite equation (76) as

$$\begin{aligned}\theta\int_t^{\infty} u(c^*(s), x^*(s))e^{-\theta(s-t)}ds = \lambda^{c^*}(t)[c^*(t) + \dot{k}^*(t) + \rho^*(t)x^*(t) \\ + \tau^*(t)\dot{x}^*(t)] + s^*(t)\end{aligned} \tag{79}$$

in which $\tau(t) = \mu^c(t)/\lambda^c(t)$. Equation (79) means that the present value of future utility (which is our welfare measure) is proportional to the sum of the linearized current value Hamiltonian and the consumer surplus. The linearized current value Hamiltonian is, in turn the real comprehensive NNP times the marginal utility value of capital. For the economy set out here, the comprehensive NNP contains four parts. The first two terms represent the conventional NNP, the third term reflects the marginal value of pollution as a consumption good (bad), and the fourth term represents the marginal value of additions to the stock of pollution (the net investment aspect of the environment). In general, the real comprehensive NNP does not constitute an exact real welfare measure due to the appearance of the consumer surplus in equation (79). However, in the special case where the instantaneous utility function is linear homogenous, $s=0$, meaning that the real comprehensive NNP is proportional to the present value of future utility.

Weitzman’s somewhat surprising result means that welfare at time t can be measured solely by using information referring to time t, although the welfare concept itself (the present value of future utility) is fundamentally intertemporal. What is the intuition behind this result? We start by observing that net investment at time t is optimally adjusted to consumption according to equation (69), which tells us that the present value at time t of an extra unit of consumption equals the present value of one additional unit of capital invested at time t. The non-arbitrage conditions, given by equations (71) and (72), mean that it is unprofitable to reallocate capital over time. This is true for all kinds of capital including the stock of pollution. Moreover, the use of energy at time t is optimally adjusted by equation (70), implying that the utility value of the last unit of energy used in production just equals the present value of the future disutility created by additional pollution. In other words, the costate variables capture the future welfare effects of the actions taken today. The final piece that generates proportionality is that the time preference is constant (otherwise, it would not be possible to place the rate of time preference outside the integral in equation (76)).

To be able to give a graphical interpretation of the current value Hamiltonian, let us finally consider Figure 4, where we use the short notations $PC = c + \rho x$ and $QI = \dot{k} + [\mu^c / \lambda^c]\dot{x}$.

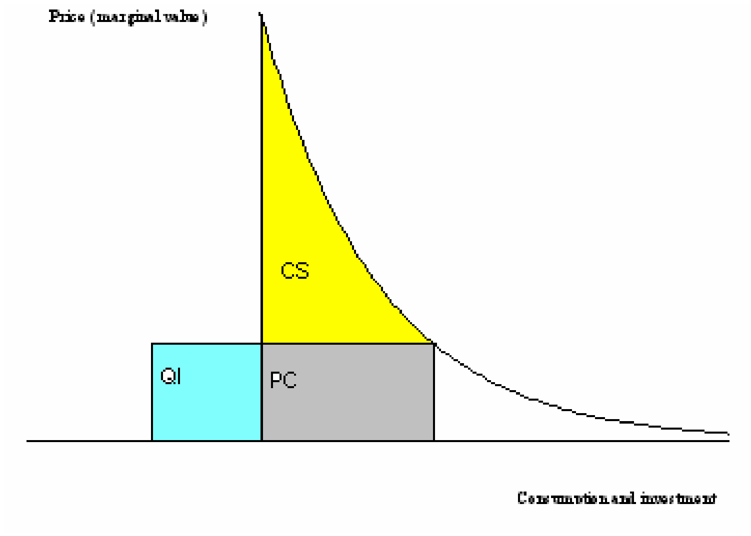

Figure 4: The Static Welfare Equivalent

In Figure 4, the area *PC* represents the producers' surplus, *CS* the consumers' surplus and *QI* the value of net investments. Therefore, the rectangle area *PC+QI* measures the value-in-exchange, i.e. the real comprehensive NNP (the linearized current value Hamiltonian divided by the marginal utility of income). Adding the consumers' surplus to real comprehensive NNP, we obtain the welfare measure.

10.3 Welfare Measurement in the Decentralized Economy

If the resource allocation is first best and the optimal control problem time-autonomous (except for the explicit time-dependence of the utility discount factor), we saw in the previous subsection that the current value Hamiltonian constitutes an exact welfare measure. In a decentralized economy, on the other hand, the resource allocation is not necessarily optimal from the perspective of society, meaning that the shadow prices may not correctly measure the future welfare consequences of the actions taken today. Therefore, Weitzman's (1976) welfare measure needs no longer apply. To illustrate, we consider a decentralized version of the model set out above, in which the externalities associated with pollution have not become internalized.

The utility maximization problem facing the consumer is given by

$$\underset{c(t)}{Max}\int_0^\infty u(c(t),x(t))e^{-\theta t}dt \qquad (80)$$

subject to

$$\dot{k}(t)=\pi(t)+r(t)k(t)+w(t)-c(t) \qquad (81)$$

as well as subject to the initial condition, $k(0)=k_0$, and a No-Ponzi Game (NPG) condition meaning that the present value of the asset (physical capital) is nonnegative at the terminal point. Equation (81) is the asset accumulation equation. The consumer supplies one unit of labor inelastically at each instant and earns labor income $w(t)$ as well as rents capital at the market rate of interest $r(t)$ to the representative firm. The term $\pi(t)\geq 0$ represents possible profit income. Note that the representative consumer treats the development of the stock of pollution as exogenous. For later use, note also that the present value Hamiltonian implicit in the consumer's decision-problem can be written as

$$H(t)=u(c(t),x(t))e^{-\theta t}+\lambda(t)\dot{k}(t) \qquad (82)$$

The representative firm would choose $k(t)$ and $g(t)$ to maximize profit at each point in time

$$\pi(t)=f(k(t),g(t))-w(t)-r(t)k(t) \qquad (83)$$

If we combine the first order conditions for the consumer and the firm, the following conditions are among those obeyed by the decentralized equilibrium (neglecting the time indicator for notational convenience);

$$u_c(c,x)e^{-\theta t}-\lambda=0 \qquad (84)$$

$$f_g(k,g)=0 \qquad (85)$$

$$\dot{\lambda}=-\lambda f_k(k,g) \qquad (86)$$

$$\lim_{t\to\infty}\lambda=0 \qquad (87)$$

There are two principal differences between the necessary conditions characterizing the decentralized economy and the first best optimal resource allocation. First, emissions are free of charge here, implying that the firm uses emissions up to a point where the marginal product of emissions is zero. Second, the stock of pollution is not an endogenous state variable in the decentralized economy; it is, instead, a side effect of the behavior of the firm and exogenous to the consumer. Let

$$\{c^0(t), g^0(t), k^0(t), x^0(t), \lambda^0(t)\}_0^\infty$$

denote the equilibrium in the uncontrolled market economy, where the government makes no attempt to reduce the negative effects of pollution.

Does the Hamiltonian governing consumer choices in the decentralized economy constitute a welfare measure? Let us carry out the same type of analysis as we did in the previous subsection. By totally differentiating the present value Hamiltonian in equation (82) with respect to time and using the necessary conditions, we obtain

$$\frac{dH^0(t)}{dt} = -\theta u(c^0(t), x^0(t))e^{-\theta t} + u_x(c^0(t), x^0(t))e^{-\theta t}\dot{x}^0(t) \tag{88}$$

The second term on the right hand side of equation (88) arises because the stock of pollution is not en endogenous state variable; it is, instead, an exogenous function of time from the perspective of the representative consumer. In other words, there is no condition to balance the marginal benefits and costs of pollution, implying a positive first order welfare effect of *t* via *x*. Solving equation (88) subject to $\lim_{t\to\infty} H^0(t) = 0$ and then transforming to current value, we obtain the welfare measure

$$\theta\int_t^\infty u(c^0(s), x^0(s))e^{-\theta(s-t)}ds = H^{c^0}(t) + M^0(t) \tag{89}$$

where

$$M^0(t) = \int_t^\infty u_x(c^0(s), x^0(s))\dot{x}^0(s)e^{-\theta(s-t)}ds$$

represents the value of the marginal externality. Note also that, if we were to reinterpret the value of the marginal externality a 'shadow price of time', i.e. as the shadow price of the artificial state variable $h(t) = t$, equation (89) may be written in the same general way as equation (76) above.

Equation (89) means that the present value of future utility (i.e. our welfare measure) is proportional to the sum of the current value Hamiltonian and the present value of the marginal externality. Note that this form of the welfare measure does not depend on the assumption that the government makes no attempt at all to internalize the externality. If the government makes such attempts, while failing to fully internalize the externality, the welfare measure will still take the same general form as in equation (89)[15], even if the magnitudes of the terms on the right hand side would change[16]. This particular form of the welfare measure should come as no surprise to the reader, since the externality is exogenous to the consumer[17]. The practical problem is, of course, that the value of the marginal externality is forward looking: the actions taken today have future welfare effects which, in the context of an uncontrolled (or imperfectly controlled) market economy are not accurately captured by the shadow prices implicit in the resource allocation. As such numbers cannot be elicited from market data, we do not envy the national income statistician whose job is to collect (or approximate) this information in practice. However, some guidance can be found in Aronsson and Löfgren (1999).

Let us now briefly return to the discussion of Pigouvian taxes in Section 7. The idea here is to introduce a dynamic analogue to the Pigouvian tax, which is such that the agents in the decentralized version of our dynamic model replicate the choices made by the social planner in subsection 10.2. In fact, the dynamic analogue to the Pigouvan tax plays two roles here; it brings the economy to the first best social optimum, and it provides useful information for accounting purposes by measuring the social opportunity cost of emissions. Recall that the externality analyzed here is caused by the use of emissions by the firm. This accumulates a stock of pollution which, in turn, tends to reduce the utility of the consumer. Therefore, if we were to impose an emission tax on the firm, which is designed to reflect the value of additions to the stock of pollution, and then repay the tax revenues lump-sum to the consumer (meaning that the budget constraint of the government balances at each instant), the externality would become internalized. To be more specific, suppose that we were to choose a sequence of emission taxes,

$$\{\tau^*(t) = -\mu^*(t)/\lambda^*(t)\}_0^\infty ,$$

where $\mu^*(t)/\lambda^*(t)$ is the real value of additions to the stock of pollution in the first best optimum (i.e. the real shadow price of $\dot{x}^*(t)$), and then impose this sequence of taxes on the firm, one can show that the decentralized equilibrium is, in fact, the first best socially optimal resource allocation; see also Tahvonen and Kuuluvainen (1993).

Note that the dynamic analogue to the Pigouvian tax is forward looking. This is seen because, if we were to solve equation (72) subject to the relevant transversality condition, we obtain

$$\mu^*(t) = \int_t^\infty u_x(c^*(s), x^*(s))^{-\theta s} e^{-\gamma(s-t)} ds$$

Therefore, the shadow price relevant for measuring the social value of additions to the stock of pollution is the present value of future marginal utilities of pollution; entities that are by no means easier to evaluate in practice than the value of the marginal externality in the decentralized economy.

For further reading about social accounting problems in distorted market economies, the reader is referred to Aronsson and Löfgren (1999), Li and Löfgren (2002), and Aronsson (1998, 2007). See also the textbook by Aronsson et al. (2004).

10.4 Briefly on Cost Benefit Analysis in Dynamic Models

The Hamiltonian is not only a useful tool for social accounting by being the comprehensive NNP in utility terms; it is also a useful tool for cost benefit analysis. Léonard (1987), Caputo (1990) and LaFrance and Barney (1991) have all given important contributions by eliciting formal cost benefit rules for parametric changes in optimal control models. Just to explain the basic idea (formal proofs are found in the aforementioned studies as well as in Aronsson et al. (1997)), note that the present value Hamiltonian, if evaluated at an equilibrium, can be written as a function of the parameters of the problem and of time itself. By using the model analyzed in the previous two subsections, and exemplifying by focusing on the first best social optimum, the optimal value function at time 0 is given by

$$V^*(0,\alpha) = \int_0^\infty u(c^*(t,\alpha), x^*(t,\alpha))e^{-\theta t}dt \qquad (90)$$

where α is any parameter (i.e. a technological parameter, a parameter characterizing an important aspect of public policy etc.), on which the initially optimal resource allocation is conditioned. The cost benefit rule for α that we were referring to can be written as

$$\frac{\partial V^*(0,\alpha)}{\partial \alpha} = \int_0^\infty \frac{\partial H^*(t,\alpha)}{\partial \alpha} dt \qquad (91)$$

in which $H^*(t,\alpha)$ is the present value Hamiltonian at time *t*, which is evaluated in the initially optimal (pre-change) resource allocation. Equation (91) is a straight forward consequence of the dynamic envelope theorem. All indirect effects of α via control, state and co-state variables vanish as a consequence of optimization, meaning that all that remains is to take the partial derivative of the Hamiltonian with respect to α (i.e. the direct effect of α) and then integrate over the planning horizon. Therefore, this is the dynamic analogue to the cost benefit rule derived in a static model in Section 3.

For further study of methodological aspects of cost benefit analysis in dynamic general equilibrium models, the reader is referred to Léonard (1987), Caputo (1990), LaFrance and Barney (1991), Aronsson et al. (1997) and Li and Löfgren (2007).

11. Final Comments and Short Summary

The idea behind this chapter has been to give an overview of modern welfare theory or, at least, a significant part thereof. We decided 'to start from the beginning' by giving the reader a historical perspective on the issues dealt with in a more formal way later on. We have not, for obvious reasons, been able to cover all relevant aspects in great detail (in fact, part of our discussion more resembles 'a scrap on the surface'), which we have tried to compensate for by giving suitable references for further study. As such, the chapter may serve as a starting point for the study of welfare economics at the graduate level. Readers who want to go deeper into specific topics ought to combine it with more specialized texts; in fact, a large such complementary literature is available.

At the core of our chapter are traditional issues and tools that any user of modern welfare theory must be aware of, such as the First and Second Welfare Theorems, Arrow's Impossibility Theorem, and situations were the markets themselves do not give rise to an optimal resource allocation from society's point of view; the latter being exemplified by externalities and public goods. As a consequence, the chapter also touches upon the normative theory of taxation. We have also taken further steps by introducing social accounting and the associated problem of measuring welfare - a growing area of research in welfare economics - as well as introduced methods for cost benefit analysis in dynamic economies. It is our hope that the reader will find this 'smorgasbord' an interesting starting point, and that it may stimulate further study of welfare theory.

References

Arrow, K (1951) *Social Choice and Individual Values*, New York: John Wileys and Sons.

Arrow, K. (1951a) *An Extension of the Basic Theorems of Welfare Economics*, Second Berkeley Symposium of Mathematical Statistics and probability, ed. J. Neyman, Berkeley: University of California press

Aronsson, T. (1998a) Welfare Measurement, Green Accounting and Distortionary Taxes. *Journal of Public Economics* **70**, 273-295.

Aronsson, T. (2007) Social Accounting and the Public Sector. Forthcoming in *International Economic Review*.

Aronsson, T., Johansson P.O. and K.G. Löfgren (1997) *Welfare Measurement, Sustainability and Green National Accounting*. Cheltenham: Edward Elgar.

Aronsson, T., Löfgren, K-G. and Backlund, K. (2004) *Welfare Measurement in Imperfect Markets – A Growth Theoretical Approach*. Cheltenham: Edward Elgar Publishing Limited.

Aronsson, T. and K.G. Löfgren (1993) Welfare Consequences of Technological and Environmental Externalities in the Ramsey Growth Model. *Natural Resource Modeling* ***7,*** 1-14.

Aronsson, T. and Löfgren, K-G. (1999b) Pollution Tax Design and Green National Accounting. *The European Economic Review* **43**, 1457-1474.

Aronsson, T. and Sjögren, T. (2004) Is the Optimal Labor Income Tax Progressive in a Unionized Economy? *Scandinavian Journal of Economics* **106**, 661-675.

Atkinson, A. and Stern, N. (1974) Pigou, Taxation and Public Goods. *Review of Economic Studies* **41**, 119-128.

Atkinson, A.B, and J.E. Stiglitz (1976) The Design of Tax Structure: Direct versus Indirect Taxation. *Journal of Public Economics* **1**, 55-75.

Backlund, K. (2003) On the Role of Green Taxes in Social Accounting. *Environmental and Resource Economics* **25**, 33-50.

Barone, E. (1908) In Ministro della Produzione nello Stato Collectivista, *Giornale degli Economisti, September/October.*

Bergson, A (1938) The Reformulation of Certain Aspects of Welfare Theory, *Quarterly Journal of Economics* ***52**, 310-34.*

Black, D. (1948) On the Rationale of Group Decision- Making, *Journal of Political Economy 56, 23-34.*

Boadway, R., and M.J. Keen (1993) Public Goods, Self-Selection and Optimal Income Taxation. *International Economic Review* **34**, 463-478.

Brock, W.A. (1977), A Polluted Golden Age, in V.L. Smith (ed.), *Economics of Natural and Environmental Resources*, New York: Gordon & Breach.

Cantillon, R (1931) *Essai sur la Nature de Commerce en General (*1755), with English translation , ed H.Higgs. London: Macmillan.

Caputo, M.R. (1990) Comparative Dynamics via Envelope Methods in Variational Calculus. *Review of Economic Studies* **57**, 689-697.

Condorcet, M.J.A.NC. (1785) *Essai sur de l'Analyse a' la Probabililite' des Decisions rendues a' laPluralite des Voix,* Paris

Christiansen, V. (1981) Evaluation of Projects under Optimal Taxation. *Review of Economic Studies* **48**, 447-457.

Clarke, E. (1971) Multipart Pricing of Public Goods. *Public Choice* **11**, 17-33.

Dasgupta, P., Hammond, P., and Maskin, E. (1999) The Implementation of Social Choice Rules, Review of Economic Studies **46,** 185-216.

Davanzati B (1965/1588) *Lezione delle Monete*. In Scittori classici italiano di economica politica, ed. P. Custodi, parte antic vol **2(**1804) Rome: Bizzarri.

Dixit, A., and Norman, V. (1980) *Theory of International Trade,* Cambridge: Cambridge University Press.

Dupuit Jules (1844 /1952) *De la Mesure de l´Utilité des Travaux Publiques* ,reprinted as On the Measurement of the Utility of Public Works, *International Economic Papers No 1952, 83-110.*

Edgeworth, F.Y. (1881) Mathematical Pshycics: *An Essay on the Application of Mathematics to Moral Sciences,* London: Keagan and Paul.

Edwards, J., Keen, M. and Tuomala, M. (1994) Income Tax, Commodity Taxes and Public Good Provision. *Finanzarchiv* **51**, 472-97.

Feldman, A (1987) *Welfare Economics, i*n the New Palgrave Dictionary of Economics, London: Macmillan.

Fisher, I. (1925/1892) *Mathematical Investigations in the Theory of Value and Prices,* New Haven: Yale University Press.

Gaube, T. (2000) When do Distortionary Taxes Reduce the Optimal Supply of Public Goods? *Journal of Public Economics* **76**, 151-180.

Gibbard, A (1973) Manipulations of Voting Schemes, A General Result, *Econometrica* **41**, 587-601

Gossen, H. (1854/1889 new ed.) *Entwicklung der Gesetze des menschlichen Verkehrs, und der daraus fliessenden Regeln¨für menschliches Handeln,* Second ed. Berlin: Prager

Groves, T. (1973) Incentives in Teams. *Econometrica* **41**, 63-80.

von Hayek, F. ed. (1935) *Collective Economic Planning,* London: Routledge.

Hurwicz, L. (1960) Optimality and Informal Efficiency in Resource Allocation Processes, in K, Arrow, S Karlin and P. Suppes, eds., *Mathematical Methods in Social Sciences,* 27-46. Stanford: Stanford University Press.

Hurwicz, L (1972) On Informationally Decentralized Systems, R. Radner and C.B.Mc Guire, eds., *Decision and Organization,* 297-336.

La France, J.T. and L.D. Barney (1991) The Envelope Theorem in Dynamic Optimization. *Journal of Economic Dynamics Control* **15**, 355-385.

Lange, O (1947) T*he Practice of Economic planning and the Optimal Allocation of Resources.* In Proceedings of the International Statistical Conference, vol 5, Washington DC

Lange, O and Taylor, F.M. (1938) (eds.) *On the Economic Theory of Socialism.* Contains an introduction by N. Lippincott, Minneapolis:University of Minnesota Press.

Law,J.(1705/1750 new ed.) *Money and Trade Considered, with a Proposal for Supplying the Nation with Money,* New ed. Glascow:Foulis.

Lerner, A.P. (1934) The Concept of Monopoly and the Measurement of Monopoly Power, *Review of Economic Studies* **1**, *157-75.*

Lerner, A.P. (1944) *The Economics of Control. Principles of Welfare Economics,* New York: Macmillan.

Leonard, D. (1987) Costate Variables Correctly Value Stocks at Each Instant. *Journal of Economic Dynamics and Control* **11**, 117-122.

Li, C.H. and Löfgren, K.G. (2002) On the Choice of Money Metrics in Dynamic Welfare Analysis: Utility versus Money Measures, Umeå Economic Studies, No 590.

Li, C-H. and Löfgren, K-G. (2006) Comprehensive NNP, Social Welfare and the Rate of Return. *Economics Letters* **90**, 254-259.

Li, C-H. and Löfgren, K-G. (2008) Evaluating Projects in a Dynamic Economy: Some New Envelope Results. *German Economic Review* **9**, 1-16

Lindahl, E. (1919) Just Taxation – A Positive Solution. In Musgrave, R. and Peacock, A. (eds) *Classics in the Theory of Public Finance*. London: Macmillan.

Maskin, E (1999) Nash Equilibrium and Welfare Optimality, working paper circulating since 1977; published in *Review of Economic Studies* **66**, *39-56.*

Michel, P. (1982) On the Transversality Conditions in Infinite Horizon Optimal Control Problems. *Econometrica* **50**, 975-985.

Mirrlees, J. (1971) An Exploration into the Theory of Optimum Income Taxation. *Review of Economic Studies* **38**, 175-208.

Mirrlees, J.A. (1976) Optimal Tax Theory: A Synthesis. *Journal of Public Economics* **6**, 327-358.

von Mises, L.E. (1935/1920) economic Calculation in the Socialist Commonwealth, in von Hayek (1935).

Myerson, R. (1979) Incentive Compatibility and the Bargaining Problem, *Econometrica **47**, 61-74.*

Nansen E.J. (1882) *Methods of Election,* Transactions and proceedings of the Royal Society of Victoria **19,** 197-240.

Niehans, J. (1990) *A History of Economic Theory,* Baltimore: Johns Hopkins.

Pareto,V. (1909) *Manuel d'Economie Politique (1906).* Trans. A. Bonnet. Paris: Giard & Briere.

Phelps, E. (1973) Taxation of Wage Income for Economic Justice. *Quarterly Journal of Economics* **87**, 331-54.

Pigou, A.C. (1912) Wealth and Welfare, London: Macmillan

Pigou, A, C. (1920) *The Economics of Welfare,* London: Macmillan.

Pigou, A.C. (1947) *A Study in Public Finance*. London: Macmillan.

Sadka, E. (1976) On Income Distribution, Incentive Effects and Optimal Income Taxation. *Review of Economic Studies* **43**, 261-267.

Samuelson, P. A. (1954). The Pure Theory of Public Expenditure. *Review of Economics and Statistics* **36**, 387-389.

Sandmo, A. (1998) Redistribution and the Marginal Cost of Public Funds. *Journal of Public Economics* **70**, 365-382.

Seierstad, A. and Sydsaeter, K. (1987) *Optimal Control Theory with Economic Applications*. Amsterdam: North-Holland.

Sen, A (1970) *Collective Choice and Social Welfare*, San Francisco: Holden-Day

Smith, A. (1876/1986) *The Wealth of Nations, Books I-III*, Reprinted as Penguin Classic 1986.

Stiglitz, J. and Dasgupta, P. (1971) Differetial Taxation, Public Goods, and Economic Efficiency. *Review of Economic Studies* **38**, 151-174.

Stern, N.H. (1982) Optimum Taxation with Errors in Administration. *Journal of Public Economics* **17**, 181-211.

Stiglitz, J.E. (1982) Self-Selection and Pareto Efficient Taxation. *Journal of Public Economics* **17**, 213-240.

Tahvonen, O. and J. Kuuluvainen (1993), Economic Growth, Pollution, and Renewable Resources, *Journal of Environmental Economics and Management* **24**, 101-118.

Walras, L. 1874-77) Elements Economique Pure. Lausanne: Rouge (reprint 1900).

Varian, H. (1994) *Microeconomic Analysis*. New York: Norton.

Warming, Jens (1911*)* Om grundrente af fiskegrunde, *Nationalökonomisk Tidskrift,* **49**, *499-505*.

Weitzman, M.L. (1976), 'On the Welfare Significance of National Product in a Dynamic Economy', *Quarterly Journal of Economics* **90**, 156-162.

Wilson, J. (1991) Optimal Public Good Provision with Limited Lump-sum Taxation *American Economic Review* **81**, 153-166.

Endnotes

[*] The authors would like to thank Pei-Chen Gong, Jason F. Shogren and Tomas Sjögren for helpful comments and suggestions.

[1] Little is known of Cantillon's early life. He was born probably between 1680 and 1690, the second son of an Irish nobleman. His most famous work is the "Essai sur la Nature du Commerce en General". The official year of publication is 1755, but this is 21 years after his death. A possible reason for this discrepancy is found in Niehans (1990). Cantillon is perhaps most famous for his insights in monetary theory and his general equilibrium theory of the three rents. An input output system more elaborated than that of Quesnay which was sketched around 1757.

[2] First mentioned by John Law (1705).

[3] Smith (1776), reprinted as Peguin Classics 1986) page 131-132.

[4] The first Italian edition appeared in 1906. The appendix of this book is best compared with Paul Samuelson, Foundations of Economic Analysis from (1947). For any utility function, $u(x)$, a monotone transformation is given by $F(u(x))$, with $F^{'} > 0$.

[5] This was not at the time, as Niehan's (1990) puts it, a revolutionary insight . It had been noted ten years later by the American Economist Irving Fisher in his thesis from 1892 published as Mathematical Investigations in the Theory of Value and Prices (1925).

[6] Pigouvian taxes was first introduced by the Dane Jens Warming (1911). He used them to solve for the first best allocation in an open access fishery model.

[7] Note that we also used the Envelope Theorem to derive equation (9) above.

[8] By substituting this expression for the subsidy into the individual budget constraint, one can see that the individual price of the public good, $(1-s)\kappa$, is equal to $MRS^{*}_{g,c}$, which is commonly referred to as the Lindahl price.

[9] See Varian (1994) for additional detail. Clearly, the system of side payments can lead to budget balance problems. Although it may not be possible to design a system where the payments sum to zero, the problem can be reduced such that the 'effective side payment' is nonpositive by supplementing the side payments with positive or negative lump-sum taxes (which do not affect the incentives at the individual level).

[10] Note that if $l_{\omega} < 0$, then $m < 1$, which means the opposite adjustment in terms of public provision.

[11] Seminal contributions to the literature on nonlinear and/or mixed taxation are Mirrlees (1971), Phelps (1973), Atkinson and Stiglitz (1976), Mirrlees (1976), Sadka (1976), Stern (1982), Stiglitz (1982) and Edwards et al. (1994). See also Christiansen (1981) and Boadway and Keen (1993) for public good provision in economies with nonlinear taxation.

[12] The intuition is that a more compressed wage distribution (i.e. a higher wage ratio, $\phi = w^1 / w^2$) discourages mimicking. We may under certain conditions reach a more compressed wage distribution by discouraging the labor supply of the low-ability type (via a higher marginal income tax rate than would otherwise be chosen) and stimulating the labor supply of the high-ability type (via a lower marginal income tax rate than would otherwise be chosen).

[13] To be more specific, solving equation (75) up to time *T* gives

$$H^*(T) = H^*(t) - \theta\int_t^T u(c^*(s), x^*(s))e^{-\theta s}ds$$

Then, using $\lim_{T\to\infty} H^*(T) = 0$, one obtains equation (76).

[14] Their contribution is to relate the sum of comprehensive NNP and the consumer surplus to a real welfare measure defined in terms of the present value of future consumption. In addition, they also connect social accounting to price index theory.

[15] See Aronsson and Löfgren (1999).

[16] See Backlund (2003) for a numerical model of social accounting, in which the empirical importance of (uniniternalized) production externalities is assessed.

[17] A similar result will be obtained if the externality is replaced by disembodied technological change. See Aronsson and Löfgren (1993).